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Strategic candidacy for multivalued voting procedures

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Abstract

Dutta et al. (Econometrica 69 (2001) 1013) (Dutta, Jackson, and Le Breton—DJLeB) initiate the study of manipulation of voting procedures by a candidate who withdraws from the election. A voting procedure is *candidate stable* if this is never possible. We extend the DJLeB framework by allowing: (a) the outcome of the procedure to be a set of candidates; (b) some or all of the voters to have weak preference orderings of the candidates. When there are at least three candidates, any strongly candidate stable voting selection satisfying a weak unanimity condition is characterized by a serial dictatorship. This result generalizes Theorem 4 of DJLeB.

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1. Introduction

The possibility that a candidate with no chance of winning might change the outcome by withdrawing from the election was a vivid feature of the 2000 US Presidential election, and is frequently a consideration when more than two

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candidates are running. Dutta, Jackson, and Le Breton (henceforth DJLeB) [14] initiate the study of "stability" with respect to such strategizing, investigating the procedures that do not allow successful manipulation of the outcome by strategic withdrawal, so that having all candidates is a stable configuration among the various possible sets of candidacies. In their framework agents may be either candidates or voters, or both, and each agent has a strict preference ordering over the set of candidates. A function assigning a feasible candidate to each preference profile-feasible set pair is, in their terminology, a *voting procedure* if it satisfies a condition we will describe as *independence of irrelevant alternatives*: the chosen candidate depends only on the voters' rankings of the feasible candidates. They only consider profiles in which each candidate prefers herself to any other candidate. In this context a natural version of *unanimity* is that a candidate is selected if she is the favorite of all voters who are not candidates and the second favorite of all other candidates who are voters.

DJLeB say that a voting procedure is *candidate stable* if there is no profile such that a candidate can achieve a preferred outcome, in comparison with the one chosen when all enter, by withdrawing. A voting procedure is *strongly candidate stable* if there is no profile such that a candidate who is not winning when all enter can change the outcome by withdrawing. If the sets of candidates and voters are disjoint, so that the candidates' preferences have no influence, independence of irrelevant alternatives implies that the two conditions are equivalent: if there is a profile allowing a candidate to influence the outcome by withdrawing, then by modifying the candidate's preferences, if need be, one may create a profile in which the candidate profits by withdrawing. From a logical point of view, the most powerful result (Theorem 4) in DJLeB is that a voting procedure satisfying unanimity and strong candidate stability is necessarily dictatorial. What they regard as their main result—that procedures satisfying unanimity and candidate stability are dictatorial when the sets of candidates and voters are disjoint—is a corollary. They give an example (with some candidates who are also voters, of course) of a nondictatorial procedure that satisfies unanimity and candidate stability.

This paper generalizes Theorem 4 of DJLeB in two directions. First, we show that it continues to hold when the mechanism may select a set of candidates with more than one element. We call such a mechanism a *voting selection*. We will say that the voting selection is *strongly candidate stable* if, for each candidate, that candidate's withdrawal from the slate of all candidates results in no change in the set specified by the selection when that candidate was not in this set, and results only in the removal of that candidate when that candidate was a member of the selected set, but not its unique member.

Our second generalization is to allow the agents to have weak preference orderings of the candidates, finding that the voting selection is characterized by serial dictatorship. That is, there is a first dictator, and the final choice is always a subset of her set of favorite candidates. The field is further winnowed by eliminating all but a second dictator's favorites from this set, all but the third dictator's favorites from the resulting set, and so forth. There is a weak preference ordering (tie breaking rule) such that the chosen set consists of the candidates that are most preferred under this ordering among those who survive all voters' vetoes.

It should be noted that our result in the case when all voters may have weak preferences does not directly imply Theorem 4 of DJLeB: the nonexistence of a function with a property, say P, on a large domain does not have implications for the existence of such a function on a small domain unless one can easily show that a function on the small domain satisfying P has an extension satisfying P. In order to obtain a framework encompassing their result and the case in which all voters may have weak preferences, we consider a mixed framework in which some voters are allowed to have weak preferences while other voters have only strict preferences. If one of the dictators always has strict preferences, after she has been consulted there will be only one remaining candidate, so the preferences of subsequent voters in the list of dictators, and the tie breaking rule, have no effect in this case. As we will see, if a candidate voter's preferences may affect the outcome, i.e., she does not come after the first dictator with strict preferences, then she must be the last in the list of dictators, since otherwise there are profiles in which dictatorship is in conflict with our version of unanimity. In particular, when a candidate voter has influence there can be only one such voter, and she is decisive only when all other voters are indifferent.

Three other recent papers have extended the concepts and analysis of DJLeB. Ehlers and Weymark [15] provide an alternative, relatively simple, proof of Theorem 4 of DJLeB. Rodríguez-Álvarez [36], which is independent, also considers the extension to multivalued voting procedures, obtaining the special case of our theorem when all agents are assumed to have strict preferences. It also provides extensive discussion of the connection between strategy proofness and possible definitions of stability for multivalued voting selections, some of which we explain below. In addition, his proof provides yet another route to Theorem 4 of DJLeB. Rodríguez-Álvarez [35] extends the analysis of DJLeB to allow outcomes that are lotteries over the set of candidates.

In addition to generalizing the result, we provide a method of proof that is significantly different from those employed by DJLeB, Ehlers and Weymark, and Rodríguez-Álvarez. These papers derive social welfare functions that are shown to satisfy the hypotheses of various versions of Arrow's theorem (Wilson [38] in the case of DJLeB; Grether and Plott [24] in the case of Ehlers and Weymark; Mas-Colell and Sonnenschein [30] in the case of Rodríguez-Álvarez [36]) leading to the conclusion that the derived social welfare functions are dictatorial, which in turn implies that the voting procedure is dictatorial. (See Section 4 for additional discussion.) Our argument is more direct, using the pivotal voter method originated by Barberà [3] and recently employed by Geanakoplos [20], Benoît [7], and Reny [34].

It is natural, both conceptually and in relation to prior literature, to be concerned with generalizing the results of DJLeB in the directions considered here. Little needs to be said to motivate the extension to weak preferences: they are natural and studied in many other models, of course. It is interesting to point out that in the context of a closely related result, the Gibbard–Satterthwaite theorem [22,37] weak preferences allow much more general notions of serial dictatorship. For example, there are nonmanipulable Paretian social choice functions given by serial dictatorships in which the second dictator may be a function of the first dictator's preference ordering, the third dictator can depend jointly on the preference orderings of the first two dictators, and so forth.

There is a large and continuing literature concerned with extensions of the Gibbard–Satterthwaite theorem to set-valued social choice functions [3,5–7, 11,13,16–18,23,25–28,36]. Many motivations have been suggested for considering set-valued social choice procedures. On the one hand, the set itself may be the final outcome of the procedure. Examples include determination of the membership of a committee, elections in which legislative representatives are selected using a system of multi-member districts, and legislative procedures in which a subset of a set of proposed pieces of legislation will eventually be passed into law [6,25–27,32].

The selected set may be the set of equilibrium outcomes of the final stage of a multistage procedure. For example, Besley and Coate [8] define a *political equilibrium* to be a subgame perfect equilibrium of a procedure in which, in the first stage, each voter decides whether to run for office. In such a setting one may think of the voting selection as specifying, for each possible set of candidates, the set of equilibrium outcomes of the subgame of the mechanism in which those candidates run. When the voting selection satisfies our version of strong candidate stability there necessarily exists a political equilibrium in which all candidates choose to run.[1]

In another large class of interpretations the selected set is the slate of candidates who survive the first stage of a multistage procedure [5,7,11,13,28,36]. The mechanism designer may or may not be able to control subsequent stages, and when the subsequent stages are not controllable, there may be more or less information about the outcomes they will produce. At one extreme, there are circumstances in which it is natural to imagine that the mechanism designer knows nothing about the process that will result in a final winner or the candidates' selection probabilities. At the other extreme the mechanism designer may know the exact selection probabilities, for instance because they are generated by the generalization of using a coin flip to select a winner in the event of the tie: the winner is selected by means of a lottery that assigns equal probability to each member of the set.

Of course different interpretations lead to different notions of manipulation and non-manipulability [19]. Our assumptions are, to a large extent, motivated by the desire to find the appropriate version of candidate stability for the interpretation in which ties are broken by predetermined lotteries. As in the strong candidate stability notion of DJLeB, we impose the requirement that if a candidate is not a member of the selected set, the selected set must be the same after that candidate withdraws. (In addition one would like to assume that the candidate's withdrawal has no impact on

[1] To construct the equilibrium we must choose a "winner" from the set specified by the voting selection for each set of candidates. The winner from the set that results when all candidates run may be chosen arbitrarily. Strong candidate stability implies that it is possible to choose the same winner after the withdrawal of any candidate other than the winner herself, and any specification of winners with this feature will be satisfactory, since the withdrawal of any candidate other than the winner does not change the outcome, and the winner prefers herself to any other candidate.

the selection probabilities of the members of the chosen set, but our framework does not encompass the information required to express this condition.) Since we assume throughout that each candidate always prefers herself to any other candidate, a candidate who is the unique element of the selected set can never achieve a more desired outcome by withdrawing. Consequently, the requirement that no candidate can manipulate by withdrawing has no implications concerning the chosen set after the candidate's withdrawal in this case, and our assumptions impose no restrictions.

The more problematic assumptions concern the impact of withdrawal when the candidate is a member of the chosen set, but not its only member. When the winner is selected by an equiprobable lottery, the requirement that no candidate can manipulate by withdrawing clearly implies a condition we call *noncontraction*: the candidate's withdrawal does not result in the removal of any other candidate from the selected set. This follows because, in the absence of information concerning the candidate's von Neumann-Morgenstern (vNM) utility function, one cannot guarantee that such a strategic withdrawal is not improving, since the withdrawing candidate might regard the eliminated candidate as disastrous. More difficult to justify is the assumption we call *nonexpansion*: the candidate's withdrawal does not result in the addition of new candidates to the chosen set. If the final winner is selected using an equiprobable lottery, and two new candidates are introduced when the candidate withdraws, this may be a successful manipulation if the new candidates are, in the withdrawing candidate's opinion, almost as good as the candidate herself. But we also impose the requirement that the effect of the candidate's withdrawal is not to replace her with a single other candidate, even though this would not create an incentive to withdraw when the final winner is chosen using an equiprobable lottery.

One justification for this assumption is that there are examples (Example 3 below and Example 1 of Rodríguez-Álvarez [36]) showing that it cannot be dispensed with. From this point of view our assumptions are not, perhaps, the most desirable, but merely the weakest that are logically possible for the types of generalizations of DJLeB that we are investigating. (Applied to voting procedures, our notion of strong candidate stability is equivalent to the DJLeB concept, so our result is a generalization, and not merely a variant, of theirs.)

Rodríguez-Álvarez [36] provides a precise justification for our definition of strong candidate stability. In his framework (following Barberá et al. [5] and Benoît [7]) each agent expresses a preference ordering over sets. Without restrictions on these orderings the sets could be taken as the primitive alternatives, so the interest in this approach arises when there are restrictions, in which case the results are not simple corollaries of the results for procedures that always choose a single candidate. As in [5], Rodríguez-Álvarez [36] focuses on the preference orderings on sets that can be derived from a vNM utility function on the candidates when a final winner is selected in one of two ways:

(a) each member of the winning set is selected with equal probability;
(b) there is a probability distribution on the set of candidates, and the selection probabilities from any subset are the derived conditional probabilities.

As Rodríguez-Álvarez [36] also points out, under (a) it will never be profitable for a candidate to withdraw if the result is to replace himself with precisely one other candidate, but under (b) it can be profitable, provided the new candidate is almost as good, in the withdrawing candidate's opinion, and is selected with higher probability. A third possibility is that the selection probabilities from the various subsets need not be derived by conditioning, but still have the property that the probability of selecting a particular candidate from some set is always greater than the probability of selecting that candidate from any proper superset. This scenario, which falls between the possibilities (a) and (b) studied by Barberá et al. [5] and Rodríguez-Álvarez [35,36] on the one hand, and the very general preferences over sets considered by Benoît [7] on the other, also justifies the more demanding form of strong candidate stability studied here.

We should mention at this point that it is possible to consider voting procedures that are explicitly probabilistic, mapping slate-profile pairs to lotteries over the slate. In such a setting Gibbard [23] shows that a social choice rule satisfying assumptions similar to those of the Gibbard–Satterthwaite theorem is necessarily a probabilistic mixture of dictatorships. One might conjecture that a probabilistic version of strong candidate stability (in conjunction with analogues of unanimity and independence of irrelevant alternatives) implies that a probabilistic voting procedure is a random dictatorship. But Pattanaik and Peleg [33, Example 5.6] give the following counterexample.

Suppose that the number of candidates is greater than two and less than the number of voters plus two. The procedure behaves like a random dictatorship except at certain profiles described below, for the slate of all candidates. Specifically, for slates that are proper subsets of the set of candidates, and for the slate of all candidates at nonexceptional profiles, the probability that a candidate is selected is proportional to the number of voters for whom that candidate is the favorite from the slate. The exceptional profiles are those in which there is a candidate that is the second choice of all voters, and every other candidate is the favorite of at least one voter. (The requirement that the number of candidates is less than the number of voters plus two guarantees that such profiles exist, and they do not have a unanimous first choice since there are at least three candidates.) For these profiles the procedure is a mixture of the equal-weights random dictatorship (with weight $1-\varepsilon$) and the unanimous second choice (with weight ε). This procedure assigns no probability to Pareto dominated candidates, so it satisfies the probabilistic version of our unanimity condition: the probability of choosing a candidate from the intersection of the voters' sets of favorites is one when that intersection is nonempty. This procedure is easily seen to have selection probabilities that depend only on voters' preferences over the slate, which is the natural probabilistic interpretation of independence of irrelevant alternatives. When $1/\varepsilon - 1$ is greater than the number of voters, withdrawal of a candidate never causes another candidate's selection probability to decrease, so withdrawal cannot be an effective manipulation.

Pattanaik and Peleg also prove a positive result that applies when there are at least two voters and the number of candidates is at least two greater than the number of voters. In this case random dictatorships are the only procedures which satisfy independence of irrelevant alternatives, assign no probability to Pareto-dominated candidates, and in which the withdrawal of a candidate does not lead to a decrease in any other candidate's selection probability. We do not know whether this result continues to hold when the probabilistic version of our unanimity condition mentioned above is substituted for their Pareto condition. Other variations of this condition are possible. For instance, one could require that, for every profile, the chosen lottery is ordinally efficient [10,12,31] by which we mean that there is no other feasible lottery that is unambiguously better for each voter in the sense of first-order stochastic dominance. Possibly because they noticed this, Pattanaik and Peleg describe their condition as "Paretian ex post."

The remainder of the paper has the following structure. In Section 2 we describe the model formally and state the main result: when there are at least two voters and three candidates unanimity and strong candidate stability imply that there is a serial dictatorship with limited powers for candidate voters. Section 3 gives several examples that show that the full strength of all assumptions are required in the theorem. Section 4 contains the proof.

2. The result

The set of individuals is $\mathscr{N}$. Individuals may be either candidates or voters. Let $\mathscr{C} \subset \mathscr{N}$ be the set of candidates and let $\mathscr{V} \subset \mathscr{N}$ be the set of voters. We assume that $\mathscr{V}$ and $\mathscr{C}$ are finite and each have at least two elements, and that $\mathscr{C} \cup \mathscr{V} = \mathscr{N}$. An important possibility is that $\mathscr{C} \cap \mathscr{V}$ is nonempty.

A weak preference relation R is a complete[2] transitive binary relation on $\mathscr{C}$. Let $\mathscr{R}_{\mathrm{w}}$ be the set of such relations. For $R \in \mathscr{R}_{\mathrm{w}}$ the associated strict preference relation P and the associated indifference relation I are defined by $aPb \Leftrightarrow [aRb \& \neg bRa]$ and $aIb \Leftrightarrow [aRb \& bRa]$, respectively. Weak preference relations on $\mathscr{C}$ will typically be denoted by $R, R', R_i, \ldots$ with $P, P', P_i, \ldots$ and $I, I', I_i, \ldots$ being the associated strict preference and indifference relations, respectively.

Let $\mathscr{R}_{\mathrm{s}}$ be the set of $R \in \mathscr{R}_{\mathrm{w}}$ that are strict in the sense that for all distinct $a, b \in \mathscr{C}$, either aPb or bPa. We allow for domains of preference profiles in which some individuals may have weak preferences while others are restricted to have strict preferences. Let $\mathscr{N}_{\mathrm{s}} \subset \mathscr{N}$ be the set of individuals who are restricted to have strict preferences, and let $\mathscr{N}_{\mathrm{w}} := \mathscr{N} \setminus \mathscr{N}_{\mathrm{s}}$ be the set of individuals who may have weak preferences. Set $\mathscr{V}_{\mathrm{s}} := \mathscr{V} \cap \mathscr{N}_{\mathrm{s}}$, and define $\mathscr{V}_{\mathrm{w}}$, $\mathscr{C}_{\mathrm{s}}$, and $\mathscr{C}_{\mathrm{w}}$ similarly.

Each candidate is assumed to prefer herself to all other candidates, but she can be indifferent between other candidates if she is in $\mathscr{N}_{\mathrm{w}}$. That is, $\mathscr{C}_{\mathrm{w}}$ may be nonempty. Thus the relevant set of profiles is $\mathscr{R} := \prod_{i \in \mathscr{N}} \mathscr{R}_i$ where, for each $i \in \mathscr{N}$,

$$\mathscr{R}_i := \{R_i \in \mathscr{R}_{\mathrm{w}} : R_i \in \mathscr{R}_{\mathrm{s}} \text{ if } i \in \mathscr{N}_{\mathrm{s}} \quad \text{and } iP_i a \text{ for all } a \in \mathscr{C} \setminus \{i\} \text{ if } i \in \mathscr{C}\}.$$

[2] R is *complete* if, for all $a, b \in \mathscr{C}$, either aRb or bRa, and possibly both. Note that this implies reflexivity: aRa for all $a \in \mathscr{C}$.

If $S \subset \mathcal{N}$ and $F \subset \mathscr{C}$, we say that profiles *R and R′ agree for S on F* if $aR_ib \Leftrightarrow aR'_ib$ for all $i \in S$ and all $a, b \in F$. We say that *R and R′ agree on F* if they agree for $\mathcal{N}$ on F. Since we are only concerned with the possibility that a candidate may choose to drop out when all candidates are running, we only consider slates of candidates that include all but at most one candidate.[3] A *roster* is a set of candidates containing all but at most one candidate; let

$$\mathscr{A} = \{A \subset \mathscr{C}: \#A \geqslant \#\mathscr{C} - 1\}$$

be the set of rosters. A *voting selection* is a function

$$V: \mathscr{A} \times \mathscr{R} \to 2^{\mathscr{C}} \backslash \{\emptyset\}$$

mapping roster-profile pairs to candidate sets with the following properties:

(i) $V(A, R) \subset A$ for all $(A, R) \in \mathscr{A} \times \mathscr{R}$;
(ii) $V(A, R) = V(A, R')$ whenever $R, R' \in \mathscr{R}$ agree for $\mathscr{V}$ on A.

We refer to condition (i) as *feasibility* and condition (ii) as *independence of irrelevant alternatives*.

A certain amount of care is involved in adapting the unanimity condition of DJLeB to our setting, where voters may have weak preferences. First of all, if there are two or more candidate voters, there can never be a unanimous favorite, and it seems natural to think that unanimity should require the choice of a candidate when she is one of the most preferred candidates of each noncandidate voters and in the second indifference class of all other candidate voters. Formally, a candidate a is a *restricted favorite from $A \in \mathscr{A}$ for individual i* with preference R_i if $a \in A$ and aR_ib for all $b \in A \backslash \{i\}$. A candidate is regarded as a *unanimous favorite* from A for a profile R if: (i) she is a restricted favorite from A for all voters; (ii) she is not weakly Pareto dominated by another candidate in A. To see the need for (ii) consider the possibility that there is one candidate voter, say a, and all other voters have both a and a's second choice b in their top indifference class. In this circumstance b is Pareto dominated, but nonetheless satisfies (i). A voting selection V satisfies *unanimity* if $V(A, R)$ is the set of unanimous favorites from A whenever this set has exactly one element. This is a weak version of the general notion of Pareto optimality. In Lemma 1 we show that, in conjunction with the other requirements on V, it implies a much stronger unanimity condition.

For a candidate $a \in \mathscr{C}$ we say that a voting selection V satisfies *a-stability* if, for all profiles R, either $V(\mathscr{C}, R) = \{a\}$ or $V(\mathscr{C} \backslash \{a\}, R) = V(\mathscr{C}, R) \backslash \{a\}$. The voting selection satisfies *strong candidate stability* if it satisfies a-stability for all $a \in \mathscr{C}$.

The concept of a-stability can be thought of as the conjunction of three conditions. We say that V satisfies *a-insignificance* if $a \notin V(\mathscr{C}, R)$ implies $V(\mathscr{C} \backslash \{a\}, R) = V(\mathscr{C}, R)$. It is said to satisfy *a-noncontraction* if $V(\mathscr{C} \backslash \{a\}, R) \supset V(\mathscr{C}, R) \backslash \{a\}$ for all $a \in V(\mathscr{C}, R)$, and it is said to satisfy *a-nonexpansion* if $V(\mathscr{C} \backslash \{a\}, R) \subset V(\mathscr{C}, R) \backslash \{a\}$ when

[3] Other authors [15,35,36] consider voting procedures that are defined for slates of candidates missing more than one candidate. One may pass by restriction from a voting procedure or selection defined on all such slates to one defined for slates missing at most one candidate, so our approach is superficially more general, but careful examination shows that their arguments do not refer to smaller slates.

$a \in V(\mathscr{C}, R)$ and a is not the unique element of $V(\mathscr{C}, R)$. We say that V *satisfies insignificance* (*noncontraction*, *nonexpansion*) if it satisfies a-insignificance (a-noncontraction, a-nonexpansion) for all $a \in \mathscr{C}$.

In our notion of serial dictatorship the voters are asked in turn, according to a given ordering, to veto all but their favorites in the set of candidates that have not yet been vetoed. If multiple candidates survive this process, then the selected set is the subset of favorites according to a given preference order, which may be thought of as a tie breaking rule. Formally, a *serial dictatorship* is a pair (π, ρ) where $\pi: \{1, \dots, \#\mathscr{V}\} \to \mathscr{V}$ is one-to-one and onto, and ρ is a weak preference on the set $\mathscr{C} \backslash \mathscr{V}$ of nonvoting candidates. For a nonempty $F \subset \mathscr{C}$ and a weak preference R let

$$top(F, R) := \{a \in F\colon\ aRb \text{ for all } b \in F\}$$

be the set of most preferred elements of F. For a serial dictatorship (π, ρ) there is an associated voting selection $V_{(\pi,\rho)}$ given by defining $V^1_{(\pi,\rho)}(A, R), \dots, V^{\#\mathscr{V}}_{(\pi,\rho)}(A, R)$ recursively, for a roster A and profile R, by

$$V^1_{(\pi,\rho)}(A, R) := top(A, R_{\pi(1)})$$

and

$$V^i_{(\pi,\rho)}(A, R) := top(V^{i-1}_{(\pi,\rho)}(A, R), R_{\pi(i)}) \quad (i = 2, \dots, \#\mathscr{V}),$$

after which we set

$$V_{(\pi,\rho)}(A, R) := top(V^{\#\mathscr{V}}_{(\pi,\rho)}(A, R), \rho).$$

We say that a voting procedure V is *dictatorial* if there is a serial dictatorship (π, ρ) such that $V(A, R) = V_{(\pi,\rho)}(A, R)$ for all $(A, R) \in \mathscr{A} \times \mathscr{R}$. For $\mathscr{R}' \subset \mathscr{R}$ we say that V is (π, ρ)-*dictatorial on* $\mathscr{R}'$ if $V(A, R) = V_{(\pi,\rho)}(A, R)$ for all $(A, R) \in \mathscr{A} \times \mathscr{R}'$.

We say that (π, ρ) is *equivalent to* (π', ρ'), and write $(\pi, \rho) \approx (\pi', \rho')$, when

$$V_{(\pi,\rho)} = V_{(\pi',\rho')}.$$

Typically, this happens because π and π' agree in their orderings of voters up to and including a voter in $\mathscr{V}_s$. We say that π is *equivalent to* π', written $\pi \approx \pi'$, if, for all $j = 1, \dots, \#\mathscr{V}$, either $\pi(j) = \pi'(j)$ or there is an integer j' with $1 \leqslant j' < j$ such $\pi(j') = \pi'(j') \in \mathscr{V}_s$. Clearly (π, ρ) is equivalent to (π', ρ') if and only if either:

(a) there is some voter in $\mathscr{V}_s$ and $\pi \approx \pi'$, or
(b) all voters are in $\mathscr{V}_w$ and $(\pi, \rho) = (\pi', \rho')$.

It turns out that the serial dictatorships (π, ρ) for which $V_{(\pi,\rho)}$ satisfies unanimity have the following property: a serial dictatorship (π, ρ) is *candidate final* if, for all j such that $1 \leqslant j < \#\mathscr{V}$, $\pi(j) \in \mathscr{C}$ implies that there is some j' such that $1 \leqslant j' < j$ and $\pi(j') \in \mathscr{V}_s$. A voting selection is *candidate final dictatorial* if it is (π, ρ)-dictatorial for a candidate final (π, ρ). Note that if $(\pi, \rho) \approx (\pi', \rho')$ and (π, ρ) is candidate final , then so is (π', ρ').

To see that $V_{(\pi,\rho)}$ does not satisfy unanimity when $\#\mathscr{C} \geqslant 3$ and (π, ρ) is not candidate final , suppose that a is the first candidate voter in the list of dictators, all earlier voters are in $\mathscr{V}_{\mathrm{w}}$, and there are other voters coming after a in the list of dictators. Consider a profile R in which all voters prior to a in the list of dictators have a and b in their top indifference set, and all voters after a in the list of dictators have b as a restricted favorite, strictly preferring b to a. Then $V_{(\pi,\rho)}(\mathscr{C}, R) = \{a\}$, but, except in one special case, b is the unique unanimous favorite. (In fact a is a unanimous favorite if and only if b is the only candidate voter after a in the list of dictators and a is b's second choice. Since $\mathscr{C}$ has three or more elements, we can choose R_b with a ranked below some third candidate.)

Our main result is:

Theorem. *Suppose $\#\mathscr{C} \geqslant 3$. Then V satisfies unanimity and strong candidate stability if and only if it is candidate final dictatorial.*[4]

A stronger form of unanimity leads to a simpler form of serial dictatorship. A voting selection V satisfies *strong unanimity* if $V(A, R)$ coincides with the intersection of all voters' sets of restricted favorites from A when this set is nonempty, even if it is not a singleton. If $V_{(\pi,\rho)}$ satisfies strong unanimity and $\mathscr{V} \subset \mathscr{N}_{\mathrm{w}}$, then ρ must be the trivial preference in which all nonvoting candidates are ranked the same, since, when two of them are unanimous favorites, the chosen set must include both. Ehlers and Weymark [15] sketch an adaptation of their methods that could be used to prove the case of the theorem pertaining to strong unanimity.

When there is only one voter, unanimity implies that the voting selection is essentially dictatorial, but, technically speaking, the theorem would be false without the assumption that there are at least two voters. Specifically consider the case of a candidate who is the only voter. Unanimity implies that this candidate will be chosen when she is a member of the roster. It also implies that the set chosen from the roster of all other candidates will be a subset of her restricted favorites, but strong candidate stability imposes essentially no restrictions on this subset. In particular, the chosen set need not be the ρ-most preferred subset of her restricted favorites for some fixed ordering ρ of the other candidates.

3. Counterexamples

We now present three examples showing that the conclusion of the theorem does not hold if we weaken our notion of strong candidate stability by dropping, respectively, insignificance, noncontraction, or nonexpansion.

Example 1 (Fixed binary agenda—FBA). Assume that $\mathscr{N}_{\mathrm{w}} = \emptyset$ and that $\#\mathscr{V}$ is odd, so that binary majority rule elections never have ties. Fix an ordering of the elements of $\mathscr{C}$ as $a_1, \ldots, a_p$. In this voting selection, applied to (A, R), where

[4] Theorem 1 of Rodríguez-Álvarez [36] is this result for the case in which all agents have strict preferences.

$A = \{a_{i_1}, \dots, a_{i_{\#A}}\}$ with $i_1 < \cdots < i_{\#A}$, a_{i_1} and a_{i_2} participate in a binary majority rule election, the one that is R-preferred by the majority is matched up against a_{i_3} in a second majority rule election, the winner in that election is matched up against a_{i_4}, and so forth. The voting selection V_{FBA} is defined by letting $V_{\text{FBA}}(A, R)$ be the singleton whose only element is the candidate that remains at the end of this sequence of elections. Unanimity and independence of irrelevant alternatives are obviously satisfied. Since a single candidate is always selected, nonexpansion and noncontraction hold trivially. If there are three candidates and three voters whose preferences constitute the Condorcet profile (that is, $a_1R_1a_2R_1a_3$, $a_2R_2a_3R_2a_1$, $a_3R_3a_1R_3a_2$) the winner of the first pairwise election loses the second election, and can change the outcome by withdrawing. Thus insignificance is violated. (Example 2 of DJLeB shows that the same conclusion is obtained when sophisticated voting[5] is used to determine the winner.)

Example 2 (Top cycle—TC). Again assume that $\mathcal{N}_{\text{w}} = \emptyset$ and that $\#\mathcal{V}$ is odd. The voting selection V_{TC} is defined by letting $V_{\text{TC}}(A, R)$ be the *top cycle* of A for profile R, which is the set of $a \in A$ such that for all $b \in A \setminus \{a\}$ there is a sequence $a = a_0, \dots, a_q = b$ in A such that for $i = 1, \dots, q$, a_i is defeated by a_{i-1} in a majority rule election. The top cycle is nonempty: if an alternative, say b, is not indirectly defeated by an alternative, say a, that indirectly defeats the maximal number of alternatives, then b must directly defeat a and indirectly defeat all alternatives indirectly defeated by a, contradicting maximality. It is obvious that V_{TC} satisfies unanimity and independence of irrelevant alternatives. Removing a candidate in the top cycle may eliminate other members of the top cycle (consider the Condorcet profile) so V_{TC} does not satisfy noncontraction. Removing a candidate a in the top cycle cannot result in new candidates joining the top cycle except when a is the only member, simply because there are fewer sequences by which a candidate not in the original top cycle might indirectly defeat a candidate in the original top cycle, so V_{TC} does satisfy nonexpansion. We claim that V_{TC} also satisfies insignificance: removing a candidate a not in the top cycle can neither expand nor contract the top cycle. To see this we begin by observing that either: (i) there is a candidate who is the unique element of the top cycle both before and after a is removed, or (ii) one may choose distinct b and c with b in the top cycle before the removal and c is in the top cycle afterwards. Then c indirectly defeats b (with or without a) and before the removal b indirectly defeats a and all other candidates, implying that c indirectly defeats all other candidates before the removal, i.e., c is in the original top cycle. If b is not in the top cycle after the removal, then b does not indirectly defeat c after the removal. But it does before the removal, so it must indirectly defeat a while a indirectly defeats c. But then a is in the top cycle before the removal, contrary to assumption.

[5] Voting is *sophisticated* if, in each binary election, each voter votes for the candidate whose victory in the current round will result in the best final outcome, given that voters in later rounds will also vote in a sophisticated manner.

Example 3 (Bidictatorship; cf. Feldman [18]). Again assume that $\mathscr{N}_{\mathrm{w}} = \emptyset$. For $i, j \in \mathscr{V} \setminus \mathscr{C}$ let V_{ij} be defined by specifying that $V_{ij}(A, R)$ is the set whose elements are the R_i-most preferred element of A and the R_j-most preferred element of A. Clearly V_{ij} satisfies unanimity, independence of irrelevant alternatives, insignificance, and noncontraction, but it does not satisfy nonexpansion because when one of the two favorites drops out, it is likely to be replaced by a new candidate. Note that V_{ij} satisfies the condition mentioned in the introduction, that the withdrawal of a candidate who is being chosen never leads to more than one new candidate being added to the chosen set.

Lemma 3 and Theorem 2 of Rodríguez-Álvarez [36] combine to imply that when there are four or more candidates, dictatorships and bidictatorships are the only strongly candidate stable voting selections satisfying insignificance, noncontraction, and the weakened form of nonexpansion that requires that a candidate's withdrawal never leads to the chosen set expanding by more than one element. (There is a similar variant of the Gibbard–Satterthwaite theorem due to Feldman [18].)

The following example (and also Example 1 of Rodríguez-Álvarez [36]) shows that when there are three candidates, there are other voting selections satisfying insignificance, noncontraction, and nonexpansion by more than one element.

Example 4 (Biased junta). Let $\mathscr{C} = \{a, b, c\}$. Assume that $\mathscr{N}_{\mathrm{w}} = \emptyset$, that $\mathscr{C} \cap \mathscr{V} = \emptyset$, and that $\mathscr{V}$ has at least three elements. Let $J \subset \mathscr{V}$ be a set with at least three elements. Then V_J is defined by specifying that $V_J(A, R)$ is the set of elements of A that are favorites of at least one member of J unless $A = \mathscr{C}$ and each candidate is the favorite of one of the voters, in which case $V_J(\mathscr{C}, R) = \{a, b\}$. Again, V_J satisfies unanimity, independence of irrelevant alternatives, insignificance, and noncontraction, and since $V_J(A, R)$ never has more than two elements, it must also satisfy nonexpansion by more than one element.

In all of the examples above we assume that $\mathscr{N}_{\mathrm{w}} = \emptyset$. In an obvious sense this strengthens the examples: the three conditions constituting candidate stability are required for the result even when all agents have strict preferences. But at the same time they raise the question of whether counterexamples exist for larger domains, e.g., the case of $\mathscr{N}_{\mathrm{s}} = \emptyset$.[6] There are diverse ways that the examples might be extended or modified to fit this case, but a detailed discussion of these would take us too far afield, so we leave the exploration of these issues to the reader.

4. Proof of the theorem

First we show that if (π, ρ) is candidate final, then $V = V_{(\pi,\rho)}$ satisfies unanimity and strong candidate stability. When there is a unanimous favorite for the profile R, any other candidate will be eliminated at some point in the successive veto procedure

[6] We would like to thank an anonymous associate editor for calling attention to this point.

defining $V_{(\pi,\rho)}$. If a is a unanimous favorite, then a is not vetoed by any noncandidate voter, and if the list of dictators ends with a candidate voter, either that voter is a herself or has been vetoed earlier in the procedure, since otherwise the candidate voter would Pareto dominate a. To see that $V_{(\pi,\rho)}$ is strongly candidate stable suppose that, for some candidate b, $V_{(\pi,\rho)}(A,R) \neq \{b\}$ and observe that, by induction,

$$V^i_{(\pi,\rho)}(\mathscr{C}\backslash\{b\},R) = top(V^{i-1}_{(\pi,\rho)}(\mathscr{C}\backslash\{b\},R),R_{\pi(i)})$$
$$= top(V^{i-1}_{(\pi,\rho)}(\mathscr{C},R),R_{\pi(i)})\backslash\{b\} = V^i_{(\pi,\rho)}(\mathscr{C},R)\backslash\{b\}$$

for all i, since $V^i_{(\pi,\rho)}(\mathscr{C},R) \neq \{b\}$, and then

$$V_{(\pi,\rho)}(\mathscr{C}\backslash\{b\},R) = top(V^{\#\mathscr{V}}_{(\pi,\rho)}(\mathscr{C}\backslash\{b\},R),\rho)$$
$$= top(V^{\#\mathscr{V}}_{(\pi,\rho)}(\mathscr{C},R),\rho)\backslash\{b\} = V_{(\pi,\rho)}(\mathscr{C},R)\backslash\{b\}.$$

We now fix a voting selection V satisfying unanimity and strong candidate stability. The remainder is devoted to the proof that V is candidate final dictatorial.

The unanimity assumption has force only when there is a high degree of consensus, insofar as the intersection of all voters' sets of restricted favorites must be a singleton in order for it to be applicable. Our first lemma shows that the conjunction of unanimity and strong candidate stability implies a much stronger version of this condition: the chosen set is a subset of any set F such that all voters prefer all elements of F to all elements of its complement (except to the extent that candidate voters in the complement of F rank themselves first) even when there is disagreement concerning which elements of F are best. In addition, Lemma 1 also extends this conclusion to situations in which one candidate (who might be well regarded) is not in the roster (Ehlers and Weymark [15, Lemma 2]) establish a stronger conclusion when all voters have strict preferences: if the roster includes a candidate a that is strictly preferred to b by all voters $i \neq b$, then b is not in the chosen set.)

Given $F \subset A \in \mathscr{A}$, let $\mathscr{R}_{(F,A)}$ denote the set of profiles R such that for each $a \in F$ and $b \in A\backslash F$, aR_ib for all $i \in \mathscr{V}\backslash\{b\}$, and there is some $i \in \mathscr{V}\backslash\{a\}$ with aP_ib. That is, each $a \in F$ weakly Pareto dominates each $b \in A\backslash F$ among voters other than b.

Lemma 1. *$V(A,R) \subset F$ for all $A \in \mathscr{A}$, all nonempty $F \subset A$, and all $R \in \mathscr{R}_{(F,A)}$.*

Proof. We argue by induction on $\#F$. If $F = \{a\}$ has one element, this element is clearly a unanimous favorite from A and is not Pareto dominated by any element of $A\backslash F$. In addition, for each $b \in A\backslash F$ there is some $i \in \mathscr{V}\backslash\{a\}$ with aP_ib, so b is not a unanimous favorite from A. Therefore the assertion follows from unanimity.

Fix F with more than one element. Assume that the claim has already been established for sets with fewer than $\#F$ elements.

We begin with the case $A = \mathscr{C}$. Consider any $R \in \mathscr{R}_{(F,\mathscr{C})}$ and any $c \in F$. Let R' be the profile obtained from R by moving c to the bottom in each individual ranking other than R_c if $c \in \mathscr{V}$. That is, R' is the unique element of $\mathscr{R}$ that agrees with R on $\mathscr{C}\backslash\{c\}$

and, for all $i \in \mathcal{N} \setminus \{c\}$, satisfies aP_ic for all $a \in \mathcal{C} \setminus \{c\}$. It is easy to see that $R' \in \mathcal{R}_{(F \setminus \{c\}, \mathcal{C})}$. (Here one must note that since $\mathcal{V}$ has more than one element, it cannot be the case that $\mathcal{V} = \{c\}$.) We may assume that $V(\mathcal{C}, R) \neq \{c\}$ since otherwise we would be done. Now either c-noncontraction or c-insignificance imply that $V(\mathcal{C}, R) \setminus \{c\} \subset V(\mathcal{C} \setminus \{c\}, R)$. But independence of irrelevant alternatives implies that $V(\mathcal{C} \setminus \{c\}, R) = V(\mathcal{C} \setminus \{c\}, R')$, and the induction hypothesis gives $V(\mathcal{C} \setminus \{c\}, R') \subset F \setminus \{c\}$. Therefore $V(\mathcal{C}, R) \subset F$, as desired.

Now suppose that $A = \mathcal{C} \setminus \{d\}$. Let R'' be the profile obtained from R by moving d to the bottom of everyone's ranking (except d's) so that R'' is the unique element of $\mathcal{R}_{(F, \mathcal{C})}$ that agrees with R on $\mathcal{C} \setminus \{d\}$ and has $bP''_i d$ for all $i \in \mathcal{N} \setminus \{d\}$ and all $b \in \mathcal{C} \setminus \{d\}$. Since there are at least two voters, there is a voter other than d, and the claim just established therefore implies that $V(\mathcal{C}, R'') \subset F$, so that in particular $d \notin V(\mathcal{C}, R'')$. Therefore independence of irrelevant alternatives and d-insignificance yield

$$V(\mathcal{C} \setminus \{d\}, R) = V(\mathcal{C} \setminus \{d\}, R'') = V(\mathcal{C}, R'') \subset F. \qquad \square$$

For $\emptyset \neq F \subset \mathcal{C}$ let $\mathcal{R}_F := \mathcal{R}_{(F, \mathcal{C})}$. The next result strengthens independence of irrelevant alternatives by showing that, for profiles in $\mathcal{R}_F$, preferences over the complement of F have no effect on the chosen set.

Lemma 2. *For all $\emptyset \neq F \subset \mathcal{C}$, if $R, R' \in \mathcal{R}_F$ agree on F, then $V(\mathcal{C}, R) = V(\mathcal{C}, R')$.*

Proof. We may assume that F is a proper subset of $\mathcal{C}$ since otherwise $R = R'$ whenever R and R' agree on F. For some $a \in \mathcal{C} \setminus F$ consider R and R' that agree on $\mathcal{C} \setminus \{a\}$. Lemma 1 implies that $a \notin V(\mathcal{C}, R)$, so $V(\mathcal{C}, R) = V(\mathcal{C} \setminus \{a\}, R)$ follows from a-insignificance. Similarly, $V(\mathcal{C}, R') = V(\mathcal{C} \setminus \{a\}, R')$, and independence of irrelevant alternatives implies that $V(\mathcal{C} \setminus \{a\}, R) = V(\mathcal{C} \setminus \{a\}, R')$, so $V(\mathcal{C}, R) = V(\mathcal{C}, R')$.

Now observe that we can pass between arbitrary $R, \tilde{R} \in \mathcal{R}_F$ that agree on F through a sequence in which adjacent pairs agree on $\mathcal{C} \setminus \{a\}$ for some $a \in \mathcal{C} \setminus F$. For example, if $\mathcal{C} \setminus F = \{c_1, c_2, \ldots, c_k\}$, then we could pass from R and $\tilde{R}$ to R' and $\tilde{R}'$ by moving c_1 to the bottom in all voters' orderings (as explained in the proof of Lemma 1) pass from these to R'' and $\tilde{R}''$ by moving c_2 to the bottom, and so forth until $R^{(k)} = \tilde{R}^{(k)}$. $\square$

The next result, which is similar to a step in the proof of Lemma 5 of DJLeB, shows that for each $a, b, c \in \mathcal{C}$ and each $R \in \mathcal{R}_{\{a,b,c\}}$, the choices made from various subsets of $\{a, b, c\}$ are "rationalizable" in the sense of conforming to a social ordering of $\{a, b, c\}$.

Lemma 3. *For any $a, b, c \in \mathcal{C}$ and any $R \in \mathcal{R}_{\{a,b,c\}}$ there is a weak preference ordering $\succcurlyeq_R$ of $\{a, b, c\}$ such that for each $F \subset \{a, b, c\}$ with $\#F \geqslant 2$,*

$$V(F \cup (\mathcal{C} \setminus \{a, b, c\}), R) = top(F, \succcurlyeq_R).$$

Proof. Fix a profile $R \in \mathscr{R}_{\{a,b,c\}}$. Define $\succcurlyeq_R$ by specifying that, for all $d, e \in \{a,b,c\}$,

$$d \succcurlyeq_R e \Leftrightarrow d \in V(\{d,e\} \cup (\mathscr{C} \backslash \{a,b,c\}), R).$$

Lemma 1 implies that

$$\emptyset \neq V(\{d,e\} \cup (\mathscr{C} \backslash \{a,b,c\}), R) \subset \{d,e\},$$

so $\succcurlyeq_R$ is complete: $d \succcurlyeq_R e$ or $e \succcurlyeq_R d$, or both.

Assume that $a \succcurlyeq_R b$ and $b \succcurlyeq_R c$. Completeness implies that there is some relabelling such that this holds.

We claim that $a \in V(\mathscr{C}, R)$. If $a \notin V(\mathscr{C}, R)$, then a-insignificance would imply that $V(\mathscr{C} \backslash \{a\}, R) = V(\mathscr{C}, R)$, so that $b \in V(\mathscr{C}, R)$ since $b \succcurlyeq_R c$. Consequently either c-nonexpansion or c-insignificance would imply that $V(\mathscr{C} \backslash \{c\}, R) \subset V(\mathscr{C}, R) \backslash \{c\}$. Since $a \succcurlyeq_R b$, $a \in V(\mathscr{C} \backslash \{c\}, R)$, so $a \in V(\mathscr{C}, R)$, a contradiction.

Now $a \in V(\mathscr{C} \backslash \{b\}, R)$ follows either from b-noncontraction or b-insignificance. That is, $a \succcurlyeq_R c$, so we have shown that $\succcurlyeq_R$ is transitive.

It remains to show that $V(\mathscr{C}, R) = top(\{a,b,c\}, \succcurlyeq_R)$. Suppose that $a \in top(\{a,b,c\})$. Above we showed that this implies that $a \in V(\mathscr{C}, R)$. (We were also assuming that $b \succcurlyeq_R c$, but the case $c \succcurlyeq_R b$ is symmetric.) Therefore $top(\{a,b,c\}, \succcurlyeq_R) \subset V(\mathscr{C}, R)$.

Now suppose that $b \notin top(\{a,b,c\}, \succcurlyeq_R)$. Suppose that $a \succ_R b$, i.e., $b \notin V(\mathscr{C} \backslash \{c\}, R)$. (Of course the case $c \succ_R b$ is symmetric.) Then $b \notin V(\mathscr{C}, R)$, since otherwise c-noncontraction or c-insignificance would imply $b \in V(\mathscr{C} \backslash \{c\}, R)$. Thus $V(\mathscr{C}, R) \subset top(\{a,b,c\}, \succcurlyeq_R)$. □

At this point the general outline of our proof strategy diverges from the methods of other authors. The thrust of the argument in DJLeB is to establish that the hypotheses of Wilson's [39] version of Arrow's Impossibility Theorem are satisfied by a derived social welfare function in certain circumstances, obtain a dictator from that result, and then show that the dictator must be the same across different applications of Wilson's result, and that the dictator's preferences govern the choice in all circumstances. The argument by Ehlers and Weymark [15] is a matter of verifying that a derived social welfare function satisfies the hypotheses of the Grether–Plott theorem, which is also a version of Arrow's theorem. In similar fashion, Rodríguez-Álvarez [36] applies a third version of Arrow's theorem due to Mas-Colell and Sonnenschein [30].

Since we need to prove that choices are governed by a serial dictatorship, appeals to standard versions of Arrow's theorem are not sufficient. It might seem more natural to appeal to the Gibbard–Satterthwaite theorem, but, as we mentioned in the introduction, in environments in which voters have weak preferences there are nonmanipulable and Paretian social choice functions that are given by more general forms of serial dictatorship than we are allowing. There is some literature concerning a serial dictatorship version of Arrow's theorem in which the social preference between two alternatives is the preference of the second dictator when the first dictator is indifferent, the preference of the third dictator when both the first and second dictator are indifferent, and so forth e.g., [1,2; 9, p. 341; 21, Theorem 3; 29,

p. 345]. To the best of our knowledge, however, there is no literature considering the possibility that a fixed tie breaking rule is used to resolve unanimous indifference, hence no existing result which would imply the theorem once a derived social welfare function had been shown to satisfy its hypotheses.

Instead of proceeding in the style of DJLeB, Ehlers-Weymark and Rodríguez-Álvarez, we adopt methods developed by Geanakoplos [20] in the context of a proof of Arrow's theorem. The general idea, which goes back to Barberà [4], is to show that a voter who is "pivotal" in a certain circumstance must also have power in various other situations, and that collectively these implications establish that the voter is a dictator. Benoît [7] applies these methods to prove a variant of the Gibbard–Satterthwaite theorem. Reny [34] gives a single pivotal voter argument that (with minor adjustments) proves both Arrow's theorem and the Gibbard–Satterthwaite theorem.

Lemma 7 shows that for any $a, b \in \mathscr{C}$ (provided there are at least three candidates) there is a serial dictatorship for profiles in $\mathscr{R}_{\{a,b\}}$. The next two results are more technical versions of this idea. Our argument is similar to Geanakoplos' insofar as we begin with a profile in which all voters prefer a and b to c, then pass through a series of profiles in which c is moved to the top one voter at a time, identifying an $\{a,b\}$-dictator by observing the point at which the choice changes.

For any $F \subset \mathscr{C}$ and any set $W \subset \mathscr{V}_{\mathrm{w}}$ of voters with weak preferences let $\mathscr{R}_F^W$ be the set of profiles $R \in \mathscr{R}_F$ in which aI_ib for all $a, b \in F$ and $i \in W$.

Lemma 4. *Suppose that* a, b, c *are distinct elements of* $\mathscr{C}$, $W \subset \mathscr{V}_{\mathrm{w}}$, *and* $\mathscr{V} \setminus (W \cup \{a,b,c\})$ *is nonempty. Then there is a voter* $i \in \mathscr{V} \setminus (W \cup \{a,b,c\})$ *such that* $a \succ_R b$ *for all* $R \in \mathscr{R}^W_{\{a,b,c\}}$ *with* aP_ib *and* $b \succ_R a$ *for all* $R \in \mathscr{R}^W_{\{a,b,c\}}$ *with* bP_ia.

Proof. Consider a profile R^0 in $\mathscr{R}^W_{\{a,b,c\}}$ in which: (i) all voters not in $W \cup \{b,c\}$ strictly prefer a to b and b to c; (ii) if $b \in \mathscr{V}$, then b strictly prefers a to c; and (iii) if $c \in \mathscr{V}$, then c strictly prefers a to b. Since there is a voter who is not in $W \cup \{a,b,c\}$, Lemma 1 implies that $V(\mathscr{C}, R^0) = \{a\}$.

We now consider a sequence of profiles $R^1, \ldots, R^{\#(\mathscr{V} \setminus W)}$ where each profile in the sequence $R^0, R^1, \ldots, R^{\#(\mathscr{V} \setminus W)}$ is obtained from the last by moving c to the top for one voter. Let $\mathscr{V} \setminus W = \{i_1, \ldots, i_{\#(\mathscr{V} \setminus W)}\}$. Supposing that R^{k-1} has been determined, we construct R^k by setting $R^k_j = R^{k-1}_j$ for all $j \in \mathscr{N} \setminus \{i_k\}$ and letting $R^k_{i_k}$ be the preference obtained from $R^{k-1}_{i_k}$ by moving c as far up as possible. If $i_k = c$, this means that $R^k_{i_k} = R^{k-1}_{i_k}$, and otherwise $R^k_{i_k}$ is the unique preference that has c most preferred (except for i_k itself, if $i_k \in \mathscr{C}$) and agrees with $R^{k-1}_{i_k}$ on $\mathscr{C} \setminus \{c\}$. Since there is a voter outside $W \cup \{a,b,c\}$, Lemma 1 implies that $V(\mathscr{C}, R^{\#(\mathscr{V} \setminus W)}) = \{c\}$, so we may let $i = i_k$ for the first k such that $c \succcurlyeq_{R^k} a$ or $c \succcurlyeq_{R^k} b$. Note that $i = c$ is impossible, since in that case $R^k = R^{k-1}$.

Lemma 3 implies that $V(\mathscr{C}, R^k) = top(\{a,b,c\}, \succcurlyeq_{R_k})$, and we claim that this set is $\{c\}$. Suppose not. Then, under $\succcurlyeq_{R^k}$, c is weakly inferior to an element of $\{a,b\}$ and weakly preferred to an element of $\{a,b\}$. We may take these two elements to be distinct, since, if there was an element, say a, that was indifferent to c, we could regard a as the one that was weakly preferred if $c \succsim_{R^k} b$ or the one that was weakly inferior if $b \succsim_{R^k} c$. So, without loss of generality, suppose that $a \succsim_{R^k} c \succsim_{R^k} b$. Except for a and b, when one or both are voters, all voters have a and b adjacent to each other, so there is a profile $\tilde{R}^k$ obtained from R^k by bringing b to the top of $\{a,b\}$, if it is not already there, for each voter in $\mathscr{V}\backslash(W \cup \{a\})$. Then R^k and $\tilde{R}^k$ agree on $\mathscr{C}\backslash\{a\}$ and $\mathscr{C}\backslash\{b\}$, and $b\tilde{P}^k_j a$ for all $j \in \mathscr{V}\backslash(W \cup \{a\})$. By independence of irrelevant alternatives, $a \succsim_{\tilde{R}^k} c \succsim_{\tilde{R}^k} b$, but (since there is a voter who is not in $W \cup \{a,b,c\}$) unanimity implies that $b \succ_{\tilde{R}^k} a$. Since this is a contradiction, we conclude that $V(\mathscr{C}, R^k) = \{c\}$, as desired.

Suppose that $aP^k_i b$. (The other case is symmetric.) Consider any profile $R \in \mathscr{R}^W_{\{a,b,c\}}$ with aP_ib that agrees with R^0 (the profile we began with) on $\mathscr{C}\backslash\{a,b,c\}$. Let $\hat{R}^k$ be the profile obtained from R^k by putting c strictly between a and b in i's preferences and changing the rankings of a and b by agents in $\mathscr{V}\backslash(W \cup \{i,a,b\})$ to agree with the ranking given by R. That is:

(i) $\hat{R}^k_j = R^k_j$ for all $j \in W \cup \{a,b\}$;
(ii) $\hat{R}^k_c = R_c$;
(iii) $d\hat{R}^k_j e \Leftrightarrow dR^k_j e$ for all $j \in \mathscr{V}\backslash(W \cup \{i,a,b\})$ and $d,e \in \mathscr{C}$ such that $\{d,e\} \neq \{a,b\}$;
(iv) $a\hat{R}^k_j b \Leftrightarrow aR_jb$ and $b\hat{R}^k_j a \Leftrightarrow bR_ja$ for all $j \in \mathscr{V}\backslash(W \cup \{i,a,b\})$;
(v) $d\hat{R}^k_i e \Leftrightarrow dR^k_i e$ for all $d,e \in \mathscr{C}$ such that $\{d,e\}$ is neither $\{a,c\}$ nor $\{b,c\}$;
(vi) $a\hat{P}^k_i c\hat{P}^k_i b$.

We claim that $a \succ_{\hat{R}^k} b$. Note that $\hat{R}^k$ and R^{k-1} agree on $\mathscr{C}\backslash\{b\}$. Our choice of k implies that $c \notin V(\mathscr{C}\backslash\{b\}, R^{k-1})$, so $a \succ_{\hat{R}^k} c$ follows from independence of irrelevant alternatives. In addition, $\hat{R}^k \in \mathscr{R}^W_{\{a,b,c\}}$ agrees with R^k on $\mathscr{C}\backslash\{a\}$. Since $V(\mathscr{C}, R^k) = \{c\}$, a-irrelevance and independence of irrelevant alternatives imply that $c \succ_{\hat{R}^k} b$. Now $a \succ_{\hat{R}^k} b$ follows from transitivity.

We now establish that $i \notin \{a,b,c\}$. As we noted above, $i = c$ is impossible. Because we are assuming that aP_kb, $i = b$ is impossible. We could have chosen R with bP_ja for all $j \in \mathscr{V}\backslash(W \cup \{a\})$, in which case $a \succ_{\hat{R}^k} b$ would be contrary to unanimity (because $\mathscr{V}\backslash(W \cup \{a,b,c\}) \neq \emptyset$) so we have $i \neq a$.

By independence of irrelevant alternatives, $V(\mathscr{C}\backslash\{c\}, R) = V(\mathscr{C}\backslash\{c\}, \hat{R}^k)$, so $a \succ_R b$. Lemma 2 implies that this conclusion holds for all $R \in \mathscr{R}^W_{\{a,b,c\}}$ with aP_ib, not just those that agree with R^0 on $\mathscr{C}\backslash\{a,b,c\}$. From symmetry it follows that $b \succ_R a$ for all $R \in \mathscr{R}^W_{\{a,b,c\}}$ with bP_ia. The proof is complete. □

Lemma 5. *Suppose that* $\#\mathscr{C} \geqslant 3$, *$a$ and b are distinct elements of* $\mathscr{C}$, *and* $W \subset \mathscr{V}_{\mathrm{w}}$. *Suppose also that* $\mathscr{V}\backslash(W \cup \{a,b\})$ *is nonempty, and if it has only one element, that voter is not a candidate. Then there is a voter* $i \in \mathscr{V}\backslash(W \cup \mathscr{C})$

such that $a \succ_R b$ *for all* $R \in \mathscr{R}^W_{\{a,b\}}$ *such that* aP_ib *and* $b \succ_R a$ *for all* $R \in \mathscr{R}^W_{\{a,b\}}$ *such that* bP_ia.

Proof. Let c be an element of $\mathscr{C} \backslash \{a,b\}$. Since c cannot be the only element of $\mathscr{V} \backslash (W \cup \{a,b\})$, Lemma 4 allows us to choose $i \in \mathscr{V} \backslash (W \cup \{a,b,c\})$ such that $a \succ_R b$ for all $R \in \mathscr{R}^W_{\{a,b,c\}}$ with aP_ib and $b \succ_R a$ for all $R \in \mathscr{R}^W_{\{a,b,c\}}$ with bP_ia. Choose $R \in \mathscr{R}^W_{\{a,b\}}$ with aP_ib. (The other case is symmetric.) Let $R' \in \mathscr{R}_{\{a,b\}}$ be the profile constructed from R by moving candidates in $\mathscr{C} \backslash \{a,b\}$ strictly below $\{a,b\}$ (except to the extent that candidate voters are constrained to prefer themselves) while preserving the ordering in the two sets. That is, R' agrees with R on both $\{a,b\}$ and $\mathscr{C} \backslash \{a,b\}$ and has each voter strictly preferring a and b to all elements of $\mathscr{C} \backslash \{a,b\}$ other than themselves. Let $R'' \in \mathscr{R}_{\{a,b\}} \cap \mathscr{R}^W_{\{a,b,c\}}$ be the profile obtained from R' by moving c to the top indifference class of voters in $W \backslash \{c\}$, and moving c below $\{a,b\}$, but above all other candidates except themselves, for voters in $\mathscr{V} \backslash (W \cup \{c\})$. In other words, R'' agrees with R' on $\mathscr{C} \backslash \{c\}$, has each voter in $W \backslash \{c\}$ indifferent between a, b, and c, and has each voter in $\mathscr{V} \backslash (W \cup \{c\})$ strictly preferring a and b to c and strictly preferring c to all elements of $\mathscr{C} \backslash \{a,b,c\}$ other than herself. Now

$$V(\mathscr{C}, R) = V(\mathscr{C}, R') = V(\mathscr{C}, R'') = \{a\}.$$

Here the first two equalities follows from Lemma 2, the third follows from Lemma 4.

It remains to show that $i \notin \mathscr{C}$. But if $i \in \mathscr{C}$, we could have chosen $c = i$ in the argument above, leading to a different i' with the asserted property. Clearly this is impossible. □

A consequence of the theorem is that there is at most one candidate voter whose preferences affect the outcome. The next result is closely related.

Lemma 6. *Suppose* $\#\mathscr{C} \geqslant 3$. *If* $\mathscr{V} \backslash \mathscr{C} \subset \mathscr{V}_{\mathrm{w}}$, *then there is at most one candidate voter.*

Proof. Set $W := \mathscr{V} \backslash \mathscr{C}$. In order to produce a contradiction suppose that $W \subset \mathscr{V}_{\mathrm{w}}$ and there are at least two candidate voters including, say, a and b. If $\mathscr{C} \backslash \{a,b\}$ had two (or more) elements, say c and d, then $\{a,b\} \subset \mathscr{V} \backslash (W \cup \{c,d\})$, so Lemma 5 would imply that there was a noncandidate voter outside W, contrary to the definition of W. Thus $\#\mathscr{C} = 3$, so suppose that $\mathscr{C} = \{a,b,c\}$ with $a, b \in \mathscr{V}$.

If $c \notin \mathscr{V}$ consider the profile R in which aP_acP_ab, bP_baP_bc, and aI_ibI_ic for all other $i \in \mathscr{V}$. Then a is the unique unanimous favorite from $\mathscr{C}$, and we must have $V(\mathscr{C}, R) = \{a\}$. Now $V(\mathscr{C} \backslash \{c\}, R) = \{a\}$ follows from c-irrelevance. But we could have begun with the profile R' in which bP'_ac, cP'_ba, and aI_ibI_ic for all other $i \in \mathscr{V}$. Reasoning as above, we find that $V(\mathscr{C} \backslash \{c\}, R') = \{b\}$. But independence of irrelevant alternatives implies that $V(\mathscr{C} \backslash \{c\}, R) = V(\mathscr{C} \backslash \{c\}, R')$. This is a contradiction.

If $c \in \mathscr{V}$ consider the profile R which embodies Condorcet profile preferences among a, b, and c while all other voters are indifferent. Specifically,

$aP_a bP_a c$, $bP_b cP_b a$, $cP_c aP_c b$, and $aI_i bI_i c$ for all other $i \in \mathscr{V}$. Unanimity implies that

$$V(\mathscr{C}\backslash\{a\}, R) = \{b\}, V(\mathscr{C}\backslash\{b\}, R) = \{c\}, V(\mathscr{C}\backslash\{c\}, R) = \{a\}.$$

For any subset of $\mathscr{C} = \{a, b, c\}$, the supposition that $V(\mathscr{C}, R)$ is this set leads quickly to a violation of strong candidate stability. For example, $V(\mathscr{C}, R) = \{a\}$ is inconsistent with b-insignificance, $V(\mathscr{C}, R) = \{a, b\}$ is inconsistent with c-insignificance, $V(\mathscr{C}, R) = \{a, b, c\}$ is inconsistent with noncontraction, and so forth. □

Lemma 6 has an implication that seems interesting, even though it is not used in our proof. We say that a voter i *has influence* if there are profiles $R, R' \in \mathscr{R}$ with $R_j = R'_j$ for all $j \neq i$ such that $V(A, R) \neq V(A, R')$ for some roster A. Let $\mathscr{V}^{\mathrm{I}}$ be the set of voters who have influence, let $\mathscr{R}^{\mathrm{I}}$ be the profiles of allowed preferences for such voters, and let *proj*: $\mathscr{R} \to \mathscr{R}^{\mathrm{I}}$ be the natural projection. One may define V^{I}: $\mathscr{A} \times \mathscr{R}^{\mathrm{I}} \to 2^{\mathscr{C}}\backslash\emptyset$ implicitly by requiring that $V^{\mathrm{I}}(A, proj(R)) = V(A, R)$ for all $(A, R) \in \mathscr{A} \times \mathscr{R}$. Clearly V^{I} satisfies feasibility and independence of irrelevant alternatives. Since strong candidate stability is a matter of comparing the outcomes for different rosters at a single profile, it is also clear that V^{I} has this property.

It also turns out that V^{I} satisfies unanimity. Suppose that the set of unanimous favorites from some $A \in \mathscr{A}$ for $R^{\mathrm{I}} \in \mathscr{R}^{\mathrm{I}}$ has exactly one element a. In order to produce a contradiction suppose that there is some $b \in V^{\mathrm{I}}(A, R^{\mathrm{I}})\backslash F$. Extend R^{I} to a profile $R \in \mathscr{R}$ by endowing each voter in $\mathscr{V}\backslash\mathscr{V}^{\mathrm{I}}$ with a preference ordering in which she strictly prefers a to all other candidates other than herself, and strictly prefers all candidates to b (unless she is b). Then the set of unanimous favorites for R from A contains $\{a\}$. It can contain b only if $\mathscr{V}\backslash\mathscr{V}^{\mathrm{I}} = \{b\}$, and all voters have both a and b as restricted favorites, and $a \in I$ so that b was not a unanimous favorite in A for R^{I} because it was Pareto dominated by a. But in this case $a, b \in \mathscr{V}$ and $\mathscr{V}\backslash\{a, b\} \subset \mathscr{V}_{\mathrm{w}}$, contrary to Lemma 6. Therefore $b \notin V(A, R)$ since V satisfies unanimity. But $V^{\mathrm{I}}(A, R^{\mathrm{I}}) = V(A, R)$, so this is a contradiction.

If the theorem holds for V^{I}, then it clearly holds for V. Therefore there is no loss of generality in assuming that all candidates have influence, but we have not found a way to take advantage of this.

Lemma 7. *Assume* $\#\mathscr{C} \geqslant 3$. *For any distinct* $a, b \in \mathscr{C}$ *there is a candidate final dictatorship* (π, ρ) *such that* V *is* (π, ρ)*-dictatorial on* $\mathscr{R}_{\{a,b\}}$.

Proof. Since there are at least two voters, Lemma 6 implies that $\mathscr{V}\backslash\{a, b\}$ cannot be empty, and if this set has exactly one element, that voter cannot be a candidate. Therefore Lemma 5 implies the existence of a voter $\pi(1) \notin \mathscr{C}$ such that $a \succ_R b$ for all $R \in \mathscr{R}^{\emptyset}_{\{a,b\}}$ with $aP_{\pi(1)}b$ and $b \succ_R a$ for all $R \in \mathscr{R}^{\emptyset}_{\{a,b\}}$ with $bP_{\pi(1)}a$. We define $\pi(2), \ldots, \pi(\#\mathscr{V})$ inductively. Assume that $\pi(1), \ldots, \pi(j-1)$ have already been determined.

If one of $\pi(1), \ldots, \pi(j-1)$ is in $\mathscr{V}_{\mathrm{s}}$, then $\pi(j)$ may be chosen arbitrarily from the remaining voters. If $j = \#\mathscr{V}$, so that there is only one remaining voter, let $\pi(j)$ be that voter. If $\pi(1), \dots, \pi(j-1)$ are all in $\mathscr{V}_w \backslash \mathscr{C}$ and there is more than one other voter, Lemma 6 implies that there must be a voter outside of $\{\pi(1), \dots, \pi(j-1)\}$ other than a and b, and if there is only one such voter, that voter cannot be a candidate. Therefore the hypotheses of Lemma 5 are satisfied. Let $\pi(j) \notin \mathscr{C}$ be the voter such that, for all $R \in \mathscr{R}_{\{a,b\}}^{\{\pi(1),\dots,\pi(j-1)\}}$, $a \succ_R b$ if $aP_{\pi(j)}b$ and $b \succ_R a$ if $bP_{\pi(j)}a$. This process continues until we choose $\pi(j) \in \mathscr{V}_s$ or there is only one remaining voter. Note that the resulting π is necessarily candidate final.

If $\mathscr{V}_s \neq \emptyset$, or if $\pi(\#\mathscr{V}) \in \{a,b\}$, an ordering ρ of $\mathscr{C} \backslash \mathscr{V}$ may be chosen arbitrarily. Otherwise consider $R \in \mathscr{R}_{\{a,b\}}$ with aI_ib for all $i = 1, \dots, \#\mathscr{V}$. Let ρ be any ordering of $\mathscr{C} \backslash \mathscr{V}$ that has a preferred to b, a and b indifferent, or b preferred to a, according to whether $V(\mathscr{C}, R)$ is $\{a\}$, $\{a,b\}$, or $\{b\}$, as is required by Lemma 1. Lemma 2 implies that the ordering of a and b under ρ is the same for all R that might be employed in this definition.

We now show that V is (π,ρ)-dictatorial on $\mathscr{R}_{\{a,b\}}$. Consider any $R \in \mathscr{R}_{\{a,b\}}$. If all voters in $\mathscr{V}$ are indifferent between a and b, then $V(\mathscr{C}, R) = V_{(\pi,\rho)}(\mathscr{C}, R)$ by virtue of the definition of ρ. Otherwise let j be the first index such that it is not the case that $aI_{\pi(j)}b$, and set $W = \{\pi(1), \dots, \pi(j-1)\}$. Without loss of generality (by symmetry) suppose that $aP_{\pi(j)}b$.

There are now two possibilities. The first is that $j = \#\mathscr{V}$, in which case

$$V(\mathscr{C}, R) = \{a\} = V_{(\pi,\rho)}(\mathscr{C}, R) \tag{1}$$

by unanimity and the definition of $V_{(\pi,\rho)}$. This equation also holds in the second case, namely that $j < \#\mathscr{V}$, since $V(\mathscr{C}, R) = \{a\}$ because Lemma 5 was used to choose $\pi(j)$, and the definition of $V_{(\pi,\rho)}$ implies that $\{a\} = V_{(\pi,\rho)}(\mathscr{C}, R)$.

Unanimity and the definition of $V_{(\pi,\rho)}$ imply that

$$V(\mathscr{C} \backslash \{a\}, R) = \{b\} = V_{(\pi,\rho)}(\mathscr{C} \backslash \{a\}, R) \tag{2}$$

and

$$V(\mathscr{C} \backslash \{b\}, R) = \{a\} = V_{(\pi,\rho)}(\mathscr{C} \backslash \{b\}, R). \tag{3}$$

By Lemma 1 and c-insignificance, (1), and the definition of $V_{(\pi,\rho)}$,

$$V(\mathscr{C} \backslash \{c\}, R) = V(\mathscr{C}, R) = V_{(\pi,\rho)}(\mathscr{C}, R) = V_{(\pi,\rho)}(\mathscr{C} \backslash \{c\}, R) \tag{4}$$

for all $c \in \mathscr{C} \backslash \{a,b\}$. The assertion we have been trying to prove consists of Eqs. (1)–(4) so the proof is complete. $\square$

We have now shown that for each two element set $F \subset \mathscr{C}$ there is a candidate final serial dictatorship (π_F, ρ_F) such that V is (π_F, ρ_F)-dictatorial on $\mathscr{R}_F$. The remaining steps in the proof are matters of showing that this conclusion can be extended to sets F with more than two elements, and that the various dictatorships are in agreement. That is, there is a single serial dictatorship that agrees with each (π_F, ρ_F) on the relevant domain.

The next lemma shows that if, for a given integer k, V is dictatorial on $\mathscr{R}_F$ for each $F \subset \mathscr{C}$ with $\#F = k$, then, in connection with sets F^* with $\#F^* = k+1$, it satisfies some of the conditions involved in being candidate final dictatorial on $\mathcal{R}_{F^*}$. This provides the key information required in the inductive extension of the conclusion to sets F^* with arbitrarily many elements.

Lemma 8. *Assume that* $\#\mathcal{C} \geqslant 3$. *Suppose that for some* $k \geqslant 2$ *it is the case that for each* F *with* $\#F = k$ *there is a candidate final dictatorship* (π_F, ρ_F) *such that* V *is* (π_F, ρ_F)*-dictatorial on* $\mathcal{R}_F$. *Then for each* F^* *with* $\#F^* = k + 1$, *each* $R \in \mathcal{R}_{F^*}$, *and any* $a \in F^*$,

$$V(\mathcal{C} \backslash \{a\}, R) = V_{(\pi_{F^* \backslash \{a\}}, \rho_{F^* \backslash \{a\}})}(\mathcal{C} \backslash \{a\}, R). \tag{5}$$

If, in addition, $a \in F^* \backslash V(\mathcal{C}, R)$, *then*

$$V(\mathcal{C}, R) = V_{(\pi_{F^* \backslash \{a\}}, \rho_{F^* \backslash \{a\}})}(\mathcal{C} \backslash \{a\}, R). \tag{6}$$

Proof. Fix F^* with $\#F^* = k + 1$ and $R \in \mathcal{R}_{F^*}$. To simplify notation we write (π_a, ρ_a) in place of $(\pi_{F^* \backslash \{a\}}, \rho_{F^* \backslash \{a\}})$. By moving a to the bottom of F^* for voters other than a we may construct $R' \in \mathcal{R}_{F^* \backslash \{a\}} \cap \mathcal{R}_{F^*}$ that agrees with R on $\mathcal{C} \backslash \{a\}$. Then

$$V(\mathcal{C} \backslash \{a\}, R) = V(\mathcal{C} \backslash \{a\}, R') = V(\mathcal{C}, R') = V_{(\pi_a, \rho_a)}(\mathcal{C}, R') = V_{(\pi_a, \rho_a)}(\mathcal{C} \backslash \{a\}, R).$$

Here the first equality is from independence of irrelevant alternatives, the second follows from Lemma 1 (which implies that $a \notin V(\mathcal{C}, R')$ since there are at least two voters, hence at least one voter other than a) and a-insignificance, and the third is by assumption. Since there are at least two voters and (π_a, ρ_a) is candidate final, $\pi_a(1) \neq a$, so the last equality follows from the definition of $V_{(\pi_a, \rho_a)}$. If $a \in F^* \backslash V(\mathcal{C}, R)$, then a-insignificance and (5) imply

$$V(\mathcal{C}, R) = V(\mathcal{C} \backslash \{a\}, R) = V_{(\pi_a, \rho_a)}(\mathcal{C} \backslash \{a\}, R). \qquad \square$$

As above, suppose that for some $k \geqslant 2$ we know that for each k-element set F, V is (π_F, ρ_F)-dictatorial on $\mathcal{R}_F$ for some candidate final serial dictatorship (π_F, ρ_F). Lemma 10 shows that for all k-element sets F, F', π_F and $\pi_{F'}$ are equivalent, but of course we should not expect to have $\pi_F = \pi_{F'}$. Similarly, there is no reason to expect ρ_F and $\rho_{F'}$ to be in any sort of agreement unless all voters have weak preferences, and even then we know nothing about how ρ_F orders pairs of nonvoting candidates that are not both in F, so the most we can hope for is that ρ_F and $\rho_{F'}$ agree on $(F \cap F') \backslash \mathcal{V}$. The following result considers the case in which $\#(F \cup F') = k + 1$.

Lemma 9. *Assume that* $\#\mathcal{C} \geqslant 3$. *Suppose that for some* $k \geqslant 2$ *it is the case that for each* F *with* $\#F = k$ *there is a candidate final dictatorship* (π_F, ρ_F) *such that* V *is* (π_F, ρ_F)*-dictatorial on* $\mathcal{R}_F$. *Then for each* $(k+1)$*-element set* F^* *and any distinct elements* $a, b \in F^*$, $\pi_{F^* \backslash \{a\}}$ *and* $\pi_{F^* \backslash \{b\}}$ *are equivalent, and if* $\mathcal{V} \subset \mathcal{N}_{\mathrm{w}}$, *then there is an ordering* ρ^* *of* $F^* \backslash \mathcal{V}$ *such that for each* $a \in F^*$, ρ^* *agrees with* $\rho_{F^* \backslash \{a\}}$ *on* $F^* \backslash (\{a\} \cup \mathcal{V})$.

Proof. Fix F^* with $\#F^* = k + 1$. To simplify notation we write π_a in place of $\pi_{F^* \backslash \{a\}}$, ρ_a in place of $\rho_{F^* \backslash \{a\}}$, etc.

Suppose, in order to produce a contradiction, that $\pi_a \not\approx \pi_b$ for some $a, b \in F^*$. Let $j \in \{1, \ldots, \#\mathscr{V}\}$ be the first index such that there exist $a, b \in F^*$ with $\pi_a(j) \neq \pi_b(j)$. (See Remark 1.) Since π_a and π_b are not equivalent, $\pi_a(1) = \pi_b(1), \ldots, \pi_a(j-1) = \pi_b(j-1)$ are all elements of $\mathscr{V}_{\mathrm{w}}$. Choosing $c \in F^* \setminus \{a, b\}$, we may assume without loss of generality that $\pi_c(j) \neq \pi_b(j)$. If $j = \#\mathscr{V}$, then $\pi_a(h) = \pi_b(h)$ for $\#\mathscr{V} - 1$ values of h, hence for all values of h since π_a and π_b are bijections, so that $\pi_a = \pi_b$. Therefore $j < \#\mathscr{V}$. Since (π_a, ρ_a) and (π_b, ρ_b) are candidate final, none of

$$\pi_a(1) = \pi_b(1) = \pi_c(1), \ldots, \pi_a(j-1) = \pi_b(j-1) = \pi_c(j-1), \pi_a(j), \pi_b(j), \pi_c(j)$$

are candidates. Therefore there is a profile $R \in \mathscr{R}_{F^*}$ in which:

(i) all voters who are not candidates strictly prefer a, b, and c to any $d \in \mathscr{C} \setminus \{a, b, c\}$, and all candidate voters strictly prefer these candidates to all candidates other than themselves;
(ii) $\pi_a(1) = \pi_b(1) = \pi_c(1), \ldots, \pi_a(j-1) = \pi_b(j-1) = \pi_c(j-1)$ are indifferent between a, b, and c;
(iii) $\pi_a(j)$ and $\pi_c(j)$ have the same preferences, preferring a to b and b to c; and
(iv) $\pi_b(j)$ prefers c to either a or b.

We have $V(\mathscr{C}, R) \subset \{a, b, c\}$ by Lemma 1. But we will show that for each subset $F \subset \{a, b, c\}$, the supposition $V(\mathscr{C}, R) = F$ entails a contradiction.

We cannot have $V(\mathscr{C}, R) = \{a, b, c\}$ since a-noncontraction would imply that

$$\{b, c\} \subset V(\mathscr{C} \setminus \{a\}, R) = V_{(\pi_a, \rho_a)}(\mathscr{C} \setminus \{a\}, R) = \{b\},$$

a contradiction. Therefore there is a $z \in \{a, b, c\} \setminus V(\mathscr{C}, R)$. Now z-insignificance and (6) imply that

$$V(\mathscr{C}, R) = V(\mathscr{C} \setminus \{z\}, R) = V_{(\pi_z, \rho_z)}(\mathscr{C} \setminus \{z\}, R).$$

Consequently $V(\mathscr{C}, R)$ must be a singleton since $\pi_z(j)$ is not indifferent between the elements of $\{a, b, c\} \setminus \{z\}$. It cannot be the case that $V(\mathscr{C}, R) = \{a\}$, since then (6) and (iv) imply that

$$V(\mathscr{C}, R) = V_{(\pi_b, \rho_b)}(\mathscr{C} \setminus \{b\}, R) = \{c\}.$$

Similarly, if $V(\mathscr{C}, R) = \{b\}$, then (6) and (iii) imply that

$$V(\mathscr{C}, R) = V_{(\pi_c, \rho_c)}(\mathscr{C} \setminus \{c\}, R) = \{a\},$$

and if $V(\mathscr{C}, R) = \{c\}$, then (6) and (iii) imply that

$$V(\mathscr{C}, R) = V_{(\pi_a, \rho_a)}(\mathscr{C} \setminus \{a\}, R) = \{b\}.$$

This contradiction completes the proof that $\pi_a \not\approx \pi_b$ is impossible.

Since $\#F^* = k + 1$, there is at least one $c \in F^* \setminus \{a, b\}$. Let R be any profile R_{F^*} with each voter indifferent between a and b and strictly preferring them to all other candidates except herself, if she is a candidate. If c and c' are both in $F^* \setminus \{a, b\}$, then $c, c' \notin V(\mathscr{C}, R)$, and $\pi_a = \pi_b$ is candidate final, so that

$$V_{(\pi_c, \rho_c)}(\mathscr{C} \setminus \{c\}, R), V_{(\pi_c, \rho_c)}(\mathscr{C} \setminus \{c\}, R) \subset \{a, b\}.$$

Thus (6) can be applied, and (6) implies that $a\rho_c b$ if and only if $a\rho_{c'} b$. Therefore we may construct a binary relation ρ^* on $F^* \backslash \mathscr{V}$ by specifying that $a\rho^* b$ if $a\rho_c b$ for $c \in F^* \backslash \{a, b\}$. This ρ^* is complete since each ρ_a is, and it remains to show that it is transitive. If $\#(F^* \backslash \mathscr{V}) \geqslant 4$ transitivity follows from the transitivity of each ρ_a, so suppose that $F^* \backslash \mathscr{V} = \{a, b, c\}$, and assume that $a\rho^* b$ and $b\rho^* c$, i.e., $a\rho_c b$ and $b\rho_a c$. If not $a\rho_b c$, so that it is not the case that $a\rho^* c$, then $V(\mathscr{C} \backslash \{b\}, R) = \{c\}$ for any profile $R \in \mathscr{R}_{\{a,b,c\}}$ in which all voters are indifferent between a, b, and c. Then the three possibilities for $V(\mathscr{C}, R)$ allowed by b-stability are $V(\mathscr{C}, R) = \{c\}$, $V(\mathscr{C}, R) = \{b\}$, and $V(\mathscr{C}, R) = \{b, c\}$. In the first case a-insignificance implies that $V(\mathscr{C} \backslash \{a\}, R) = \{c\}$, contrary to $b\rho_a c$, and in the second and third case c-insignificance and c-nonexpansion, respectively, imply that $V(\mathscr{C} \backslash \{c\}, R) = \{b\}$, contrary to $a\rho_c b$. □

Lemma 10. *Assume that $\#\mathscr{C} \geqslant 3$. Suppose that for some $k \geqslant 2$ it is the case that for each F with $\#F = k$ there is a candidate final dictatorship (π_F, ρ_F) such that V is (π_F, ρ_F)-dictatorial on $\mathscr{R}_F$. Then there is a candidate final dictatorship (π, ρ) such that for all F with $\#F = k$, π_F and π are equivalent, and if $\mathscr{V} \subset \mathscr{N}_{\mathrm{w}}$, then ρ_F and ρ agree on $F \backslash \mathscr{V}$.*

Proof. Any two k-element subsets of $\mathscr{C}$ are connected by a sequence of k and $k+1$ element set obtained from alternating additions and deletions of single elements. Therefore Lemma 9 implies that $\pi_F \approx \pi_{F'}$ for all k-element sets F, F'. Let π be any representative of the equivalence class containing π_F for all k-element sets F. Lemma 9 also implies that if $\mathscr{V} \subset \mathscr{N}_{\mathrm{w}}$, then any two nonvoting candidates must be ordered in the same way by ρ_F for all k-element sets F that contain both. Therefore we may let ρ be the binary relation on $\mathscr{C} \backslash \mathscr{V}$ defined by requiring that $a\rho b$ if and only if $a\rho_F b$ for all k-element sets F that contain both. For any $(k+1)$-element set F^*, ρ agrees with the associated ρ^* on $F^* \backslash \mathscr{V}$ given by Lemma 9, so, since $k \geqslant 2$, ρ is transitive. Clearly (π, ρ) is candidate final, since each (π_F, ρ_F) is. □

In view of Lemma 7, the following result implies that V is candidate final dictatorial, and thus completes the proof of the theorem. Its proof is a matter of inductively extending the domain of the serial dictatorship to the sets of profiles $\mathscr{R}_F$ for sets F of all cardinality.

Lemma 11. *Assume that $\#\mathscr{C} \geqslant 3$. Suppose that for each F with $\#F = 2$ there is a candidate final dictatorship (π_F, ρ_F) such that V is (π_F, ρ_F)-dictatorial on $\mathscr{R}_F$. Then V is candidate final dictatorial.*

Proof. By Lemma 10 there is a candidate final (π, ρ) such that for each $F \subset \mathscr{C}$ with $\#F = 2$, V is (π, ρ)-dictatorial on $\mathscr{R}_F$. We argue by induction: assuming that for some k such that $2 \leqslant k < \#\mathscr{C}$, V is (π, ρ)-dictatorial on $\mathscr{R}_F$ for all F with $\#F = k$, we will show that this also holds with k replaced by $k+1$. Fixing $F^* \subset \mathscr{C}$ with $\#F^* = k+1$, our goal is to show that V is (π, ρ)-dictatorial on $\mathscr{R}_{F^*}$.

Fix $R \in \mathscr{R}_{F^*}$. We first show that $V(\mathscr{C}, R) = V_{(\pi,\rho)}(\mathscr{C}, R)$. Lemma 1 implies that $V(\mathscr{C}, R) \subset F^*$. If $V(\mathscr{C}, R) = F^*$, then, for each $a \in F^*$, a-nonexpansion, a-noncontraction, and (5) of Lemma 8 imply that

$$F^* \backslash \{a\} = V(\mathscr{C} \backslash \{a\}, R) = V_{(\pi,\rho)}(\mathscr{C} \backslash \{a\}, R),$$

so that all voters and ρ are indifferent between all elements of $F^* \backslash \{a\}$. Since F^* has at least three elements, this argument, applied to each $a \in F^*$, implies that all voters and ρ are indifferent between all elements of F^*, so that $V(\mathscr{C}, R) = F^* = V_{(\pi,\rho)}(\mathscr{C}, R)$, as desired.

Now suppose that $V(\mathscr{C}, R)$ is a proper subset of F^*. Fix $a \in F^* \backslash V(\mathscr{C}, R)$. We wish to show that $V(\mathscr{C}, R) = V_{(\pi,\rho)}(\mathscr{C}, R)$, and from (6) of Lemma 8 we have $V(\mathscr{C}, R) = V_{(\pi,\rho)}(\mathscr{C} \backslash \{a\}, R)$, so it suffices to show that $V_{(\pi,\rho)}(\mathscr{C} \backslash \{a\}, R) = V_{(\pi,\rho)}(\mathscr{C}, R)$. Since F^* has at least three elements there is some $b \in F^*$, $b \neq a$, such that $V(\mathscr{C}, R) \neq \{b\}$. Lemma 8(5) implies that $V(\mathscr{C} \backslash \{b\}, R) = V_{(\pi,\rho)}(\mathscr{C} \backslash \{b\}, R)$, and b-nonexpansion or b-insignificance implies that $V(\mathscr{C} \backslash \{b\}, R) \subset V(\mathscr{C}, R) \backslash \{b\}$, so that $a \notin V_{(\pi,\rho)}(\mathscr{C} \backslash \{b\}, R)$. The definition of $V_{(\pi,\rho)}$ now implies both that $a \notin V_{(\pi,\rho)}(\mathscr{C}, R)$ and, therefore, that $V_{(\pi,\rho)}(\mathscr{C} \backslash \{a\}, R) = V_{(\pi,\rho)}(\mathscr{C}, R)$, as desired.

The final step is to show that $V(\mathscr{C} \backslash \{a\}, R) = V_{(\pi,\rho)}(\mathscr{C} \backslash \{a\}, R)$ for all $a \in \mathscr{C}$. When $a \in F^*$ this follows from (5) of Lemma 8. For $a \notin F^*$ we have

$$V(\mathscr{C} \backslash \{a\}, R) = V(\mathscr{C}, R) = V_{(\pi,\rho)}(\mathscr{C}, R) = V_{(\pi,\rho)}(\mathscr{C} \backslash \{a\}, R),$$

where the first equality is from Lemma 1 and a-insignificance, the second is the result just established, and the third is from the definition of $V_{(\pi,\rho)}$. $\square$

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References

[1] F.T. Aleskerov, A.V. Vladimirov, Hierarchical voting, Inform. Sci. 39 (1986) 41–86.
[2] T.E. Armstrong, Hierarchical arrow social welfare functions, Econ. Theory 2 (1992) 27–41.
[3] S. Barberà, The manipulation of social choice mechanisms that do not leave 'too much' to chance, Econometrica 45 (1977) 1573–1588.
[4] S. Barberà, Pivotal voters: a new proof of Arrow's theorem, Econ. Lett. 6 (1980) 13–16.
[5] S. Barberà, B. Dutta, A. Sen, Strategy-proof social choice correspondences, J. Econ. Theory 101 (2001) 374–394.
[6] S. Barberà, H. Sonnenschein, L. Zhou, Voting by committees, Econometrica 59 (1991) 595–609.
[7] J.-P. Benoît, Strategic manipulation in voting games when lotteries and ties are permitted, J. Econ. Theory 102 (2001) 421–436.
[8] T. Besley, S. Coate, An economic model of representative democracy, Quart. J. Econ. 112 (1997) 85–114.
[9] C. Blackorby, D. Donaldson, J.A. Weymark, Social choice with interpersonal utility comparisons: a diagrammatic introduction, Int. Econ. Rev. 25 (1984) 327–356.
[10] A. Bogomolnaia, H. Moulin, A new solution to the random assignment problem, J. Econ. Theory 100 (2001) 295–328.
[11] S. Ching, L. Zhou, Multi-valued strategy-proof social choice rules, Soc. Choice Welfare 13 (3) (2002) 569–580.
[12] H. Crès, H. Moulin, Scheduling with opting out: improving upon random priority, Oper. Res. 49 (4) (2001) 565–577.
[13] J. Duggan, T. Schwarz, Strategic manipulability without resoluteness or shared beliefs: Gibbard–Satterthwaite generalized, Soc. Choice Welfare 17 (2000) 85–93.
[14] B. Dutta, M.O. Jackson, M. Le Breton, Strategic candidacy and voting procedures, Econometrica 69 (2001) 1013–1037.
[15] L. Ehlers, J.A. Weymark, Candidate stability and nonbinary social choice, Econ. Theory 22 (2003) 233–243.
[16] A. Feldman, Nonmanipulable multi-valued social decision functions, Public Choice 34 (1979) 177–188.
[17] A. Feldman, Manipulation and the Pareto rule, J. Econ. Theory 21 (1979) 473–482.
[18] A. Feldman, Strongly nonmanipulable multi-valued collective choice rules, Public Choice 35 (1980) 503–509.
[19] P. Gärdenfors, On definitions of manipulation of social choice functions, in: J.J. Laffont (Ed.), Aggregation and Revelation of Preferences, North-Holland, Amsterdam, 1978.
[20] J. Geanakoplos, Three brief proofs of Arrow's theorem, Cowles Foundation Discussion Paper 1123RRR, Cowles Foundation, Yale University, 2001.
[21] L. Gevers, On interpersonal comparability and social welfare orderings, Econometrica 47 (1973) 75–89.
[22] A. Gibbard, Manipulation of voting schemes: a general result, Econometrica 41 (1973) 587–601.
[23] A. Gibbard, Manipulation of schemes that mix voting and chance, Econometrica 45 (1977) 665–681.
[24] D. Grether, C. Plott, Nonbinary social choice: an impossibility theorem, Rev. Econ. Stud. 49 (1982) 143–149.
[25] B.-G. Ju, A characterization of strategy-proof voting rules for separable weak orderings, Soc. Choice Welfare 21 (2003) 469–499.
[26] B.-G. Ju, Plurality, Pareto efficiency, and strategy-proofness, Mimeo, University of Kansas, 2003.
[27] B.-G. Ju, A characterization of plurality-like rules based on non-manipulability, restricted efficiency, and anonymity, Mimeo, University of Kansas, 2003.
[28] J.R. Kelly, Strategy-proofness and social choice functions without singlevaluedness, Econometrica 45 (1977) 439–446.
[29] R.D. Luce, H. Raiffa, Games and Decisions, Wiley, New York, 1958.
[30] A. Mas-Colell, H. Sonnenschein, General possibility theorems for group decisions, Rev. Econ. Stud. 39 (1972) 185–192.

[31] A. McLennan, Ordinal efficiency and the separating hyperplane theorem, J. Econ. Theory 105 (2002) 435–449.
[32] E. Miyagawa, Locating libraries on a street, Soc. Choice Welfare 18 (3) (2001) 527–541.
[33] P.K. Pattanaik, B. Peleg, Distribution of power under stochastic social choice rules, Econometrica 54 (1986) 909–921.
[34] P.J. Reny, Arrow's theorem and the Gibbard–Satterthwaite theorem: a unified approach, Econ. Lett. 70 (2001) 99–105.
[35] C. Rodríguez-Álvarez, Candidate stability and probabilistic voting procedures, Mimeo, University of Warwick, 2003.
[36] C. Rodríguez-Álvarez, Candidate stability and voting correspondences, Mimeo, University of Warwick, 2003.
[37] M.A. Satterthwaite, Strategy-proofness and arrow's conditions: existence and correspondence theorems for voting procedures and social welfare functions, J. Econ. Theory 6 (1975) 187–217.
[38] R. Wilson, Social choice without the Pareto principle, J. Econ. Theory 5 (1972) 478–486.