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Uniqueness of stationary equilibrium payoffs in coalitional bargaining ☆

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Abstract

We study a model of sequential bargaining in which, in each period before an agreement is reached, the proposer's identity is randomly determined, the proposer suggests a division of a pie of size one, each other agent either approves or rejects the proposal, and the proposal is implemented if the set of approving agents is a winning coalition for the proposer. The theory of the fixed point index is used to show that stationary equilibrium expected payoffs of this coalitional bargaining game are unique. This generalizes Eraslan [34] insofar as: (a) there are no restrictions on the structure of sets of winning coalitions; (b) different proposers may have different sets of winning coalitions; (c) there may be a positive probability that no proposer is selected.

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Keywords: Noncooperative bargaining; Multilateral bargaining; TU games; Simple games; Coalitions; Legislative bargaining; Fixed points; Fixed point index

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1. Introduction

Baron and Ferejohn [13] study a model in which a group of n risk neutral agents divide a fixed pie. In each period a *proposer* (or "formateur" in a legislative context) is selected randomly, the proposer suggests a division of the pie, and this division is implemented if it is approved by a *winning coalition* of the agents. Otherwise the process is repeated until agreement is achieved, with payoffs discounted geometrically.

As with many other game theoretic models, useful conclusions and tractable empirical methodologies can be precluded, or at least impaired, by significant multiplicity of equilibria. Baron and Ferejohn show that there can be a wide variety of nonstationary equilibria, so any hope for uniqueness depends on restricting attention to stationary equilibria. Except in special cases there will also be a multiplicity of stationary equilibria. Roughly, an agent's expected payoff at the beginning of a period is the sum of what she expects when she is a proposer, which is the surplus left over after she has purchased the cooperation of a minimal cost coalition, plus her discounted continuation value times the probability that another proposer pays for her cooperation. Holding each agent's overall probability of being included in some other agent's coalition fixed, the pairwise probabilities of inclusion can be highly indeterminate. For this reason one cannot hope to have uniqueness of stationary equilibrium.

The remaining hope, then, is that the equilibrium expected payoffs may be determinate. Baron and Ferejohn show that if the model is symmetric in the sense that all agents have the same *recognition probability* (probability of being selected as the proposer) and discount factor, and a winning coalition is any set of k agents, then in all stationary subgame perfect equilibria agreement is reached in the first period with probability one and each agent's ex ante expected utility is $1/n$.

For the application motivating Baron and Ferejohn (bargaining among parties in a legislature or parliament) it is natural to suppose that recognition probabilities differ across agents, with larger parties typically having higher recognition probabilities. In a committee one would normally expect that the chair's recognition probability is higher than the recognition probabilities of other members. For several years it was unknown whether there could be multiple stationary subgame perfect equilibria yielding different expected utilities when recognition probabilities or discount factors differ across agents. Eraslan [34] resolved this problem, showing that, even with unequal recognition probabilities and discount factors, there is a single vector of expected utilities common to all stationary subgame perfect equilibria. Her analysis is restricted to k-majority rule for $1 \leqslant k \leqslant n$, but in legislative settings it is also natural to allow different agents to have different weights in the voting over approval of a proposal. This direction of generalization is also of interest from the point of view of other applications. In corporate bankruptcies governed by Chapter 11, the voting over approval of a proposed reorganization is asymmetric with respect to different seniority classes of debt, and creditors who are owed more money have greater power. Other examples are described in the next section.

Here we show that, under more general conditions than those considered by Eraslan [34], there is a unique vector of expected payoffs that is generated by all of the game's stationary subgame perfect equilibria. Specifically, in addition to allowing different agents to have different recognition probabilities and discount factors, we allow the set of winning coalitions to be arbitrary, and to depend on the proposer, and we allow the sum of the recognition probabilities to be less than one.

To a certain extent Eraslan's result, and our generalization, are intuitive: holding an agent's expected surplus when she is the proposer fixed, her expected payoff is a strictly increasing function of the probability that some agent pays for her cooperation. In addition, agreement is reached in the first period that this is possible, so the sum of expected payoffs is constant. If we compare two different possible continuation value vectors with this sum, the overall tendency should be in the direction of more expensive coalitions being chosen less frequently by proposers. Intuitively, one expects that the adjustment dynamics in a neighborhood of an equilibrium are stable—in the one dimensional case a higher price engenders lower demand—and in fact formal analogues of this intuition are embodied in our analysis of the local properties of equilibrium. However, there are complications arising from the fact that changing the vector of continuation values changes the vector of proposer values and (especially in our case) from the complexity of the structure of winning coalitions.

The passage from local analysis to global uniqueness has an interesting mathematical structure. Suppose that C is a nonempty compact convex set and $F : C \to C$ is an upper semicontinuous convex valued correspondence. Roughly, the *fixed point index* assigns an integer to each compact set of fixed points of F that has a neighborhood containing no other fixed points. For any partition of the set of fixed points into such sets, the sum of the indices of the sets must be one. We show that each connected component of the set of fixed points of the relevant correspondence has a neighborhood that has no other fixed points, and that its index is one. Consequently the set of fixed points must consist of a single connected component. We also show that the vector of continuation payoffs is constant in each connected component, so our main result follows. As has been noted (cf. p. 615 of Mas-Colell et al. [58]) this method of proving uniqueness is widely applicable, and its applicability to the style of model analyzed here has been studied by Kalandrakis [50], but in all earlier cases we know of elementary methods were also available. So far as we know this is also the first application of index theory in economics in which the connected components of the set of equilibria need not be singletons, even for generic parameters. In this sense it stands in contrast to the literature on uniqueness and maximal number of Nash equilibria (e.g., Kreps [54], McKelvey and McLennan [61], McLennan [63], von Stengel [92]) which is exclusively devoted to the generic case in which all equilibria are isolated. Other works studying the structure of the set of stationary equilibria of a stochastic game include Haller and Lagunoff [40], which establishes generic finiteness for the standard stochastic game model of Shapley [87], and Haller and Lagunoff [41], which establishes generic finiteness for the two person asynchronous choice model.

We now describe the organization of the remainder. Section 2 describes some of the extensive literature descended from Baron and Ferejohn [13] as it relates to our work.

Section 3 describes the bargaining model, and defines a "reduced equilibrium" concept that is the focus of the rest of the paper. A reduced equilibrium consists of a vector of equilibrium expected payoffs and a matrix whose ij-entry is the probability that agent j includes agent i in the minimal winning coalition she forms when she is the proposer. Lemma 1 gives a characterization of the set of reduced equilibria for a given vector of equilibrium payoffs. Using this, we state the main result, Theorem 1, whose gist is that all reduced equilibria have the same vector of expected payoffs.

The description of the model in Section 3, and the motivation of reduced equilibrium in terms of stationary subgame perfect equilibrium of our bargaining model, is somewhat informal. Appendix A gives a precise formal description of the bargaining protocol, and proves that every stationary subgame perfect equilibrium has an associated reduced equilibrium, and that every reduced equilibrium is derived from some stationary subgame perfect equilibrium.

Proposition 2 in Section 4 establishes that the set of reduced equilibria is the set of fixed points of a correspondence satisfying the hypotheses of Kakutani's theorem. Section 5 presents the axioms that characterize the fixed point index. The main consequences are: (a) the index of a closed set of fixed points, that has a neighborhood without any other fixed points, depends only on the correspondence in a neighborhood of the set; (b) for any partition of the set of fixed points into such sets, the sum of the indices of the sets is one. The set of reduced equilibria consists of finitely many connected components; the large scale strategy of the proof of Theorem 1 is to show that an arbitrary component has index one, so there can be only one component, and that the vector of expected payoffs is the same at all points in the component.

The proof of Theorem 1 has two main parts. In Section 5 we define a notion of an "attracting set" of fixed points. Theorem 2, whose proof encompasses Proposition 5 and Lemmas 4–7, asserts that if each fixed point is contained in an attracting set, then the set of fixed points is a single attracting set. The second part, which is presented in Section 6 and encompasses Lemmas 8–12, is the verification that an arbitrary reduced equilibrium is contained in an attracting set.

Some possible topics for further research are sketched in Section 7.

2. Related literature

Since Baron and Ferejohn's paper, an extensive body of work has grown out of their model.[1] There are many variants of the basic model, and many ways in which the bargaining model might be embedded into some periods of larger dynamic models. The number and diversity of applications in the literature suggest that the methodology pioneered by Baron and Ferejohn [13] has established itself as an important tool for addressing a central issue of political science: the relationship between the rules governing political institutions and the outcomes they produce.

Perhaps the main alternatives would be the power indices of cooperative game theory. In Section 7 we describe how our result opens the way to similar power indices with explicit non-cooperative foundations.

As in other types of social scientific modeling, models with unique (or perhaps finitely many) predictions are preferred for many reasons. As a practical matter, tractable empirical methodologies typically limit attention to models with unique predictions because available statistical methods have little to say about selection of an equilibrium when more than one are available. Indeed, Drouvelis et al. [33] use the Baron–Ferejohn model as the basis of experimental research, and there are already several studies (e.g., Diermeier et al. [30], Diermeier and Merlo [32], Ansolabehere et al. [3], Coscia [24], Adachi and Watanabe [1]) taking the Baron–Ferejohn model to data. Our result has direct application to several previous papers (e.g., Winter [94], McCarty [60], Ansolabehere et al. [4], Snyder et al. [90]) either strengthening the work by providing uniqueness results that were not available to the authors, or allowing uniqueness to be proved under weaker

[1] With apologies for the inevitable omissions, a fairly comprehensive list is: Baron [9], Baron [10], McKelvey and Riezman [62], Rausser and Simon [84], Okada [76], Merlo and Wilson [66], Baron [11], Calvert and Dietz [19], Okada [77], Winter [94], Chari et al. [21], Persson [80], Baron [12], Diermeier and Feddersen [31], Banks and Duggan [5], McCarty [59], McCarty [60], Persson et al. [81], Bennedsen and Feldmann [16], Eraslan [34], Eraslan and Merlo [35], Jackson and Moselle [44], Norman [75], Yan [95], Ansolabehere et al. [4], Cho and Duggan [23], Diermeier et al. [30], Diermeier and Merlo [32], Coscia [24], Knight [53], Snyder et al. [90], Banks and Duggan [6], Kalandrakis [49], Kalandrakis [50], Battaglini and Coate [14], Cardona and Ponsati [20], Montero [70], Predtetchinski [82], Yıldırım [97], Adachi and Watanabe [1], Battaglini and Coate [15], Miyakawa [68], Yan [96], Breitmoser [18], Drouvelis et al. [33], Fan et al. [37], Herings and Predetchinski [43], Montero [71], Nohn [73], Nohn [74], Proost and Zaporozhets [83], Tsai and Yang [91], Yıldırım [98], Okada [78], Le Breton et al. [55]. In addition, there are many papers that consider models with more complicated state spaces (e.g., a repeatedly redetermined status quo) or are in other ways somewhat more distantly related.

hypotheses. In addition, the literature contains models (e.g., McCarty [59]) that would become instances of our framework after small modifications. It seems natural to expect that further development of the literature will lead to additional applications, and that our uniqueness result will influence model selection in some instances; indeed, Montero [70] and Le Breton et al. [55] have already applied our result in an analysis of the Council of Ministers of the European Union, and Proost and Zaporozhets [83] has applied it to Belgian railway investment.

The introduction has already described some of the history of uniqueness results for the Baron–Ferejohn model. In addition to Eraslan [34], papers concerned with uniqueness of equilibrium expected payoffs include Norman [75], Yan [95], Cho and Duggan [23], Kalandrakis [50], Montero [69], Cardona and Ponsati [20], and Yan [96]. Norman [75] shows that equilibrium payoffs may fail to be unique when there are finitely many bargaining periods. Cho and Duggan [23], Cardona and Ponsati [20], and Herings and Predetchinski [43] consider models in which the space of outcomes is one dimensional, modeling policy concerns rather than private rewards. Rausser and Simon [84] is similar, except that the issue space is multidimensional and uniqueness holds for generic utility functions. Cho and Duggan [23] establish uniqueness when utility functions are quadratic and provide an example with multiple equilibria when the utility functions are not quadratic. Kalandrakis [50] is closely related to our work because index theory is applied. In particular, Theorem 6 shows that there is one equilibrium when the determinant of the Jacobian of the relevant system of equations has the same sign at every equilibrium, and it is pointed out that this gives a proof of uniqueness for a special case of the Merlo and Wilson [66] model with unanimity rule. Kalandrakis [51] applies a two dimensional version of index theory in the context of the model of Eraslan [34], providing an algorithm that computes equilibrium values and giving an alternative version of her uniqueness result. Cardona and Ponsati [20] establish uniqueness in the case of unanimity and asymptotic uniqueness as the discount factor goes to one. Yıldırım [97,98] analyzes models in which recognition probabilities are determined by agents' efforts, either in each period (for unanimity rule and k-majority rule) or persistently (for unanimity rule) in the sense that effort in the initial period determines a single vector of recognition probabilities governing the process in all subsequent periods.

Recall that a *cooperative game with transferable utility* (TU game) consists of a set of agents $N = \{1, \dots, n\}$ together with a specification of a payoff $v(S) \in \mathbb{R}$ for each coalition $S \subseteq N$. Generalizing results in Okada [76] that did not appear in Okada [77], Yan [95] studies the bargaining protocol analyzed here, with general recognition probabilities, applied to a TU game with a nonempty core. She shows that an allocation in the core is realized as the vector of continuation values if and only if it coincides with the vector of recognition probabilities, in which case there are no other stationary subgame perfect equilibrium payoffs.

A TU game is said to be *simple* if each coalition's payoff is either zero or one. That is, a simple game is essentially a specification of a system of winning coalitions. Since we study simple games, which rarely have a nonempty core, the overlap of our results with those of Yan is small. A TU game is *proper* if it is simple and $v(S) = 1$ implies that $v(T) = 0$ for all $T \subseteq N \setminus S$. It is easy to see that if a TU game is simple and has a nonempty core, then it must be proper. Montero [69] generalizes Yan's theorem to games that are not proper, as follows: if the vector of recognition probabilities is in the nucleus[2] and the sum of recognition probabilities over the members of each winning coalition is at least $1/2$, then the vector of recognition probabilities is

[2] The nucleus is the set of imputations minimizing the maximum, over all coalitions, of the difference between the coalition's worth and its aggregate allocation in the imputation. It coincides with the core when the core is nonempty, and the nucleolus (cf. Schmeidler [86]) is always an element.

the unique vector of stationary subgame perfect equilibrium payoffs. The uniqueness assertion is a special case of our result. Okada [78] extends some of this analysis to a model in which bargaining among the remaining players continues after an agreement by a coalition other than the grand coalition.

Following Okada [76,77], Yan [96] studies a model that is more general than ours, insofar as the game need not be simple, and also less general than ours, insofar as all agents have the same recognition probability and discount factor. She shows that stationary subgame perfect equilibrium expected payoffs are unique when the game is symmetric in the sense that the worth of a coalition depends only on the number of members. She says that an equilibrium is "inclusive" if, for each agent and each coalition that agent proposes to with positive probability, proposing to a proper superset of that coalition yields a strictly lower surplus. She shows that if the game is convex (that is, for any coalitions S and T, the sum of the worths of S and T is not greater than the sum of the worths of $S \cap T$ and $S \cup T$) then all inclusive stationary subgame perfect equilibria have the same expected utilities.

Merlo and Wilson [66] and Eraslan and Merlo [35] study a generalization of the Baron–Ferejohn model in which the size of the pie varies stochastically; Merlo and Wilson [66] demonstrate uniqueness under unanimity rule, and Eraslan and Merlo [35] demonstrate nonuniqueness under majority rule. Ali [2] examines a variant of the Baron–Ferejohn model in which agents have divergent, and optimistic, beliefs about the recognition probabilities, finding that there are unique continuation payoffs even though agreement need not be reached in the first period.

3. The model and result

The agents in the set $N := \{1, \dots, n\}$ bargain over the division of a pie of size 1 according to the following protocol. At the beginning of each period until agreement is reached there is a random determination of whether there will be a *proposer* and, if so, who that will be. The probability that agent i is selected to be the proposer, denoted by p_i, is called i's *recognition probability*. Let $p := (p_1, \dots, p_n)$ be the vector of recognition probabilities. Of course we require that each p_i is nonnegative, and also that $p_1 + \dots + p_n \leqslant 1$. Then $p_0 := 1 - (p_1 + \dots + p_n)$ is the probability that there is no proposer.

If there is a proposer she selects a proposal from the set

$$\Pi := \left\{ x \in [0, 1]^n \colon \sum_{i=1}^{n} x_i \leqslant 1 \right\}$$

of feasible allocations. There is a random determination of an ordering of the agents, after which the agents each vote for or against the proposal, with each agent seeing the votes of other agents in the ordering before selecting her own vote. For each i there is a collection $\mathcal{S}_i = \{S_{i1}, \dots, S_{iK_i}\}$ of subsets of N, called *winning coalitions for* i. If i is the proposer and the set of agents voting in favor is an element of $\mathcal{S}_i$, then the proposal is implemented, ending the game. Otherwise the process is repeated in the next period. As a matter of convention we assume that i is a member of every coalition in $\mathcal{S}_i$. The utility for agent i if the proposal x is implemented in period t is $\delta_i^t x_i$ where $\delta = (\delta_1, \dots, \delta_n) \in (0, 1)^n$ is a vector of discount factors. If agreement is never reached, then each agent's utility is zero.

In Appendix A we define a notion of stationary subgame perfect equilibrium for this bargaining game. We are interested in the expected payoffs resulting from such equilibria.

Remark. Baron and Ferejohn [13] give a method for constructing an example in which the set of payoffs associated with nonstationary subgame perfect equilibria game is quite large. The underlying intuition is that if history matters, one can induce a wide variety of behavior by threatening to punish deviators, so this construction depends on remembering more than the current date. We do not know whether there can be subgame perfect equilibria of the bargaining game whose strategies are Markov, which is to say that behavior depends on the current state and calendar time, but not stationary, so that the dependence on time is nontrivial. Such an equilibrium would correspond to a sequence $(v_0, Y_0), (v_1, Y_1), \ldots$ satisfying (a) and (b) in the definition of reduced equilibrium below, with Y replaced by Y_t (except on the right hand side of (b), where Y' has no date) and v replaced by v_t on the left hand side of (a) and by v_{t+1} elsewhere. In fact we do not even know whether there can be a nontrivial cyclic equilibria, e.g., with strategies depending on the day of the week.

We now define a "reduced" equilibrium concept that gives a recursive description of the restrictions on stationary subgame perfect equilibrium payoffs. For each i and $S \in \mathcal{S}_i$ let $\tilde{S} \in \{0,1\}^n$ be the vector whose jth component is 1 if $j \in S$ and 0 otherwise. Let $\mathcal{Y}_i \subseteq [0,1]^n$ be the convex hull of $\{\tilde{S}\colon\ S \in \mathcal{S}_i\}$, and let

$$\mathcal{Y} := \mathcal{Y}_1 \times \cdots \times \mathcal{Y}_n,$$

regarded as a space of $n \times n$ matrices Y whose ith column y_i is the element of $\mathcal{Y}_i$. Then each $Y \in \mathcal{Y}$ has entries in $[0,1]$ and ones on the diagonal. Let

$$\mathcal{V} := \left\{ v \in [0,1]^n\colon\ \sum_i v_i \leqslant 1 \right\};$$

elements of $\mathcal{V}$ are thought of as possible expected payoff vectors of the game.

Definition 1. A *reduced equilibrium* is a pair $(v, Y) \in \mathcal{V} \times \mathcal{Y}$ such that for each i:

(a) $v_i = \mathbf{m}_i(Y)\delta_i v_i + p_i \mathbf{w}_i(v, y_i)$ where

$$\mathbf{m}_i(Y) := p_0 + \sum_{j=1}^n p_j y_{ij} \quad \text{and} \quad \mathbf{w}_i(v, y_i) := 1 - \sum_{j=1}^n y_{ji}\delta_j v_j;$$

(b) $y_i \in \operatorname{argmin}_{y'_i \in \mathcal{Y}_i} \sum_{j=1}^n y'_{ji}\delta_j v_j$.

This concept has the following intuitive motivation. Prior to the beginning of the initial period, agent i has an expected payoff for the game, which is denoted by v_i. This consists of two parts:

(i) the discounted value $\delta_i v_i$ of the game beginning in the next period times:
  (α) the probability p_0 that no proposer is selected, plus;
  (β) the sum $\sum_{j=1}^n p_j y_{ij}$ over all agents (including herself) of that agent's recognition probability times the probability that agent i is included in that proposer's coalition;

(ii) the recognition probability p_i times the surplus $\mathbf{w}_i(v, y_i)$ net of payments to coalitions members (including herself) when she is the proposer.

If she is asked to join another proposer's coalition, she will refuse to approve the proposal if offered less than $\delta_i v_i$, and will accept if offered this much or more. If she is the proposer she will make a proposal that consists of offering $\delta_j v_j$ (or, intuitively, slightly more to insure acceptance) to each member of one of her winning coalitions, the rest of the pie to herself, and zero to everyone else. In particular, equilibrium is efficient in the sense that agreement is reached in the first period in which there is a proposer. Each agent has a probability distribution over winning coalitions when she is the proposer, and of course this distribution assigns positive probability only to those coalitions that minimize expenditures on coalition partners.

In Appendix A we show that if any superset of a winning coalition for an agent is also a winning coalition for that agent, then each stationary subgame perfect equilibrium induces a reduced equilibrium, and each reduced equilibrium is induced by at least one stationary subgame perfect equilibrium. This result is technically demanding, but unsurprising (the voting protocol was carefully designed with these results in mind) and its primary purpose is to provide one noncooperative foundation for our interest in reduced equilibrium.

Although the intuitive justification for the reduced equilibrium concept depends on supersets of winning coalitions being winning coalitions, the definition itself, and the subsequent analysis, does not. It is, at least hypothetically, possible that there is a derivation of this concept from noncooperative analysis that does not depend on that assumption.

Turning to the analysis of reduced equilibrium, let $\mathbf{m} : \mathcal{Y} \to [0, 1]^n$ be the function whose ith component is $\mathbf{m}_i(Y)$.

Lemma 1. *If (v, Y) is a reduced equilibrium and $Y' \in \mathcal{Y}$, then (v, Y') is a reduced equilibrium if and only if $\mathbf{m}(Y') = \mathbf{m}(Y)$.*

Proof. First suppose that $\mathbf{m}(Y') = \mathbf{m}(Y)$. Then Y' minimizes total expenditure by all proposers on coalition partners. It follows that each y_i minimizes proposer i's expenditure on coalition partners, i.e., Y' satisfies (b). Therefore $\mathbf{w}_i(v, y_i') = \mathbf{w}_i(v, y_i)$, so that (a) is also satisfied. Now suppose that (v, Y') is a reduced equilibrium. Then (b) implies that $\mathbf{w}_i(v, y_i') = \mathbf{w}_i(v, y_i)$ for all i, and from (a) it follows that $\mathbf{m}_i(Y') = \mathbf{m}_i(Y)$. □

Our main result characterizes the set of reduced equilibria.

Theorem 1. *The set of reduced equilibria is nonempty, and there is a single vector $v \in \mathcal{V}$ that is the first component of every reduced equilibrium. If (v, Y) is a reduced equilibrium, then the set of all reduced equilibria is $\{v\} \times \mathbf{m}^{-1}(\mathbf{m}(Y))$. In particular, the set of reduced equilibria is a convex polytope.*

The lemma above implies that the set of reduced equilibria is $\bigcup \{v\} \times \mathbf{m}^{-1}(v)$ where the union is over all v such that there is some Y such that (v, Y) is a reduced equilibrium. The rest of our analysis is concerned with showing that there is a unique such v.

Remark. We saw above that for each agent the crucial characteristics of an equilibrium are the minimal cost of forming a winning coalition when she is the proposer and the aggregate probability of being included in some agent's winning coalition. The latter probabilities can be achieved by various assignments of probability to the proposers' winning coalitions, so the probabilities that the proposers assign to winning coalitions are indeterminate, and it is natural to try to reduce the analysis to conditions on the agents' aggregate probabilities of being included. Unfortunately this does not work, because it does not include enough information to analyze the minimal cost of a winning coalition. Instead our analysis operates at an intermediate level of aggregation, in the sense that we study the probabilities that each proposer assigns to each possible coalition partner.

4. Reduced equilibria as fixed points

This section describes a correspondence whose fixed points correspond to reduced equilibria. For the time being we fix $v \in \mathbb{R}^n$ and $Y \in \mathcal{Y}$. Our first goal is to show that if condition (a) of the definition of reduced equilibrium holds, then v is determined by Y.

We begin by introducing some notational conventions that will simplify the algebra to come. If $\nu \in \mathbb{R}^n$, $\nabla(\nu)$ will denote the $n \times n$ diagonal matrix whose diagonal entries are the components of ν. Often we will denote such a diagonal matrix with the capital letter corresponding to the lower case letter denoting the vector. Thus $P := \nabla(p)$ and $\Delta := \nabla(\delta)$. For $Y \in \mathcal{Y}$ let $M(Y) := \nabla(\mathbf{m}(Y))$.

Fix $Y \in \mathcal{Y}$. When Y is given we usually write m and M in place of $\mathbf{m}(Y)$ and $M(Y)$. Let I be the $n \times n$ identity matrix. Condition (a) in the definition of reduced equilibrium can be rewritten as

$$v = \left(M - PY^T\right)\Delta v + p$$

and then as $Av = p$ where, for $Y \in \mathcal{Y}$,

$$A = A(Y) := I - \left(M - PY^T\right)\Delta.$$

Proposition 1. *A is invertible.*

In preparation for the proof, and for a key result later, we quickly review the theory of nonsingular M-matrices. A square matrix is a *nonsingular M-matrix* if it has positive entries on the diagonal and nonpositive off-diagonal entries, and is dominant diagonal, meaning that for each column, the sum of the entries is positive. The main properties of these matrices are as follows:

Lemma 2. *If B is a nonsingular M-matrix, then B is invertible, and the entries of its inverse are nonnegative. The determinants of the principal minors are nonnegative, so if B is symmetric, then it is positive definite.*

Proof. E.g., Theorem 2.3 of Chapter 6 of Berman [17]. □

Let $A = (a_{ij})$. Then

$$a_{ij} = \begin{cases} 1 - (m_i - p_i)\delta_i, & j = i, \\ p_j y_{ij} \delta_i, & j \neq i. \end{cases}$$

Lemma 3. *$B := A - p\delta^T$ is a nonsingular M-matrix.*

Proof. Let $B = (b_{ij})$. Then $b_{ii} = 1 - m_i\delta_i > 0$. Also, for $j \neq i$ we have $b_{ji} = p_j(y_{ij} - 1)\delta_i$, and of course this is nonpositive because $y_{ij} \leqslant 1$. Therefore

$$-\sum_{j \neq i} b_{ji} = \left(\sum_{j \neq i} p_j - \sum_{j \neq i} p_j y_{ij}\right)\delta_i = \left(\sum_{j=1}^n p_j - \sum_{j=1}^n p_j y_{ij}\right)\delta_i$$
$$= (1 - m_i)\delta_i < b_{ii}. \qquad \square$$

Proof of Proposition 1. In view of the last result, the entries of B^{-1} are nonnegative, so $1 + \delta^T B^{-1} p > 0$, and consequently the Sherman–Morrison formula[3] implies that $A = B + p\delta^T$ is invertible. □

Now that Proposition 1 is established, for $Y \in \mathcal{Y}$ we may define

$$\mathbf{v}(Y) := A(Y)^{-1} p.$$

For each i and $v \in \mathcal{V}$ let

$$\mathbf{Y}_i(v) := \operatorname{argmin}_{y_i \in \mathcal{Y}_i} \sum_{j=1}^{n} y_{ji} \delta_j v_j = \operatorname{argmin}_{y_i \in \mathcal{Y}_i} y_i^T \Delta v,$$

and let

$$\mathbf{Y}(v) := \mathbf{Y}_1(v) \times \cdots \times \mathbf{Y}_n(v).$$

Let $F : \mathcal{Y} \to \mathcal{Y}$ be the correspondence $F(Y) := \mathbf{Y}(\mathbf{v}(Y))$.

Proposition 2. *The correspondence F is upper semicontinuous and convex valued. A point $Y \in \mathcal{Y}$ is a fixed point of F if and only if $(\mathbf{v}(Y), Y)$ is a reduced equilibrium.*

Proof. Since each $\mathbf{Y}_i(v)$ is the subset of $\mathcal{Y}_i$ that minimizes a linear function, $\mathbf{Y}(v)$ is compact and convex, so F is convex valued. The minimization problem varies continuously with v, and $\mathbf{v}(Y)$ is a continuous function of Y, so Berge's theorem of the maximum implies that F is upper semicontinuous.

Suppose $Y \in \mathcal{Y}$ and $v = \mathbf{v}(Y)$. Then condition (a) in the definition of a reduced equilibrium is satisfied, and condition (b) is equivalent to $Y \in \mathbf{Y}(v)$. Therefore Y is a fixed point of F if and only if (v, Y) is a reduced equilibrium. □

For $Y \in \mathcal{Y}$ and $v \in \mathbb{R}^n$ let

$$\langle\langle Y, v \rangle\rangle := \sum_i y_i^T \Delta v.$$

Then (v, Y) is a reduced equilibrium if and only if $v = \mathbf{v}(Y)$ and $\langle\langle Y' - Y, v \rangle\rangle \geqslant 0$ for all $Y' \in \mathcal{Y}$. Thus we see that the problem we are studying is a *variational inequality*, which is a problem of the form "find $x \in C$ such that $\langle f(x), x' - x \rangle \geqslant 0$ for all $x' \in C$" where C is a convex subset of $\mathbb{R}^k$ and $f : C \to \mathbb{R}^k$ is a function. Variational inequalities have an extensive literature, and have many forms of economic and game theoretic equilibria as special cases; Jofré et al. [45] provides a survey with special emphasis on aspects of interest to mathematical economics. The function f is (strictly) *monotone* if $\langle f(x) - f(x'), x - x' \rangle$ $(>)$ $\geqslant 0$ for all (distinct) $x, x' \in C$. If x^* and x^{**} are solutions of the problem, then

$$\langle f(x^*) - f(x^{**}), x^* - x^{**} \rangle \leqslant 0,$$

[3] If B is a nonsingular $n \times n$ matrix, $u, v \in \mathbb{R}^n$ are column vectors, and $\lambda := v^T B^{-1} u \neq -1$, then the formula $(B + uv^T)^{-1} = B^{-1} - B^{-1} uv^T B^{-1}/(1 + \lambda)$ can be verified by multiplying the right hand side by $B + uv^T$. (Cf., p. 124 of Meyer [67].)

from which it follows that all solutions of a monotone variational inequality give the same value, and a strictly monotone variational inequality has at most one solution. The intuition behind our analysis is in this direction, but our problem is only monotone in a certain local sense, so instead we use the fixed point index, as described in the next section. Simsek et al. [89] studies the application of the fixed point index to variational inequalities.

An issue of interest, here and in general, is how the solution and value of a variational inequality change as the parameters of the problem vary. Dafermos [26] provides general methods for studying this issue, which could potentially be applied to our problem if condition (iv) of the definition of an attracting set (in the next section) could be strengthened along the lines of her condition (1.3). Our problem has the special feature that the probabilities that various coalitions form may be indeterminate, which makes it hard to represent the vector of continuation values as the solution of a system of parameterized equations near a generic parameter vector. The function mapping the discount factors and recognition probabilities to the equilibrium vector of values has a semi-algebraic[4] graph, by virtue of the Tarski–Seidenberg theorem (e.g., Theorem 2.3 of Coste [25]) so decomposition results for semi-algebraic sets (e.g., Section 2.3 of Coste [25]) imply that smooth comparative statics are possible in principle on a generic subset of the space of parameters, but we do not study the problem herein.

5. The general uniqueness result

This section explains the general mathematical principle underlying the uniqueness asserted in Theorem 1. We first review the relevant results concerning the fixed point index, then describe the general class of correspondences for which we are able to show that the set of fixed points has a single connected component.

Between Brouwer's (1910) proof of his fixed point theorem and the middle of the last century, there emerged a theory of a fixed point index that assigns an integer to each closed set of fixed points of a suitable correspondence that is *isolated*, in the sense of having a neighborhood containing no other fixed points. Let M be a compact smooth manifold and let $f : M \to M$ be a smooth function, each of whose fixed points p is regular in the sense that

$$\mathrm{Id}_{T_p M} - Df(p) : T_p M \to T_p M$$

is nonsingular. (Here $T_p M$ is the space of tangent vectors at p, e.g., Section 1.2 of Guillemin and Pollack [38], $\mathrm{Id}_{T_p M}$ is the identity function, and $Df(p)$ is the derivative of f at p.) The index of such a p is the sign of the determinant of this linear transformation, and the index of a set of fixed points (which must be finite because each fixed point is isolated) is the sum of the indices of its members. Extending to more general settings (functions that are merely continuous, correspondences, infinite sets of fixed points) entails various complications, but the central idea is to approximate with the smooth case.

Fix a nonnegative integer m and a set $X \subseteq \mathbb{R}^m$ that is a retract of an open set.[5] An *index admissible correspondence* for X is an upper semicontinuous convex valued correspondence $F : \overline{U} \to X$ where $U \subseteq X$ is open, its closure $\overline{U}$ is compact, and F has no fixed points in $\partial U := \overline{U} \setminus U$. (When X is a manifold with boundary, or is a subset of a larger "ambient" space,

[4] A subset of a Euclidean space is *semi-algebraic* if it is a finite union of sets defined by conjunctions of polynomial equations and inequalities. Coste [25] provides a relatively brief introduction to semi-algebraic geometry.

[5] That is, there is an open V containing X and a continuous function $r : V \to X$ with $r(x) = x$ for all $x \in X$. Such an X is a *Euclidean neighborhood retract.*

it is important to bear in mind a clear distinction between the boundary of X and the boundary of U in the relative topology of X.) Let $\mathcal{C}$ be the set of index admissible correspondences.

Proposition 3. *(See McLennan [65], McLennan [64].) There is a unique function $\Lambda : \mathcal{C} \to \mathbb{Z}$ satisfying the following conditions*:

(A) (*Normalization.*) *If $c : \overline{U} \to X$ is a constant function whose constant value lies in U, then $\Lambda(c) = 1$.*
(B) (*Additivity.*) *If $F : \overline{U} \to X$ is index admissible, $U_1, \ldots, U_r \subseteq U$ are open and pairwise disjoint, and $\overline{U} \setminus (U_1 \cup \cdots \cup U_r)$ contains no fixed points of F, then*

$$\Lambda(F) = \sum_{i=1}^{r} \Lambda(F|_{\overline{U}_i}).$$

(C) (*Continuity.*) *If $F : \overline{U} \to X$ is index admissible, then there exists a neighborhood $\mathcal{V} \subseteq \overline{U} \times X$ of the graph of F such that $\Lambda(F') = \Lambda(F)$ whenever $F' : \overline{U} \to X$ is an upper semicontinuous convex valued correspondence whose graph is contained in $\mathcal{V}$.*

The following is a key property of the index, which is the basis of various results in economics and game theory asserting that the number of equilibria is generically odd. (E.g., Wilson [93], Dierker [29], Harsanyi [42].) More generally, for any partition of the set of fixed points into isolated sets, the sum of the indices of the sets must be one.

Proposition 4. *If $\overline{U} \subset X$ is convex and $F : \overline{U} \to \overline{U}$ is in $\mathcal{C}$, then $\Lambda(F) = 1$.*

Since any two continuous functions from $\overline{U}$ to itself are homotopic,[6] because $\overline{U}$ is convex, for functions this result follows from Normalization and Continuity because the latter condition implies that the index is constant along a homotopy; the smooth case is illustrated on p. 598 of Mas-Colell et al. [58]. The main difficulty in the proof (e.g., Kakutani [48]) is to show that any upper semicontinuous convex valued correspondence can be approximated by a continuous function.

We now introduce the framework of the general uniqueness result, which is slightly more general than the framework of the last two sections. Let

$$\langle\langle \cdot, \cdot \rangle\rangle : \mathbb{R}^m \times \mathbb{R}^k \to \mathbb{R}$$

be a given bilinear pairing, let $\mathcal{Y} \subset \mathbb{R}^m$ be a compact, convex polytope, and for $v \in \mathbb{R}^k$ let

$$\mathbf{Y}(v) := \operatorname{argmin}_{Y \in \mathcal{Y}} \langle\langle Y, v \rangle\rangle.$$

Let $\mathbf{v} : \mathcal{Y} \to \mathbb{R}^k$ be a C^1 function, and let $F : \mathcal{Y} \to \mathcal{Y}$ be the correspondence

$$F(Y) := \mathbf{Y}\big(\mathbf{v}(Y)\big).$$

The argument used to prove Proposition 2 shows that this correspondence also satisfies the hypotheses of Kakutani's fixed point theorem.

[6] If X and Y are topological spaces, two continuous functions $f, g : X \to Y$ are *homotopic* if there is a continuous function (called a *homotopy*) $h : X \times [0, 1] \to Y$ such that $h(\cdot, 0) = f$ and $h(\cdot, 1) = g$.

We will say that $\mathcal{E} \subseteq \mathcal{Y}$ is an *attracting set* if it is nonempty and (i)–(iv) below are satisfied.

(i) $\mathbf{v}$ is constant on $\mathcal{E}$.

Let v be the constant value of $\mathbf{v}$ on $\mathcal{E}$.

(ii) $\mathcal{E} \subseteq \mathbf{Y}(v)$.

Note that, by (i) and (ii), $\mathcal{E}$ is contained in the set of fixed points of F.

(iii) $\mathcal{E}$ is the intersection of $\mathcal{Y}$ with an affine subspace of $\mathbb{R}^m$.

Let S be the affine hull of $\mathbf{Y}(v)$, and let K be the linear subspace of $\mathbb{R}^m$ that is parallel to S. Let T be the affine hull of $\mathcal{E}$, and let L be the linear subspace of $\mathbb{R}^m$ that is parallel to T. Concretely, K (L) is the span of all vectors of the form $Y' - Y''$ where $Y', Y'' \in \mathbf{Y}(v)$ ($Y', Y'' \in \mathcal{E}$). For any $Y \in \mathcal{E}$ we have $S = Y + K$, $T = Y + L$, and since $\mathcal{E} \subset \mathbf{Y}(v)$, it follows that $T \subseteq S$ and $L \subseteq K$. Evidently $\mathbf{Y}(v) \subseteq S \cap \mathcal{Y}$, and the affine functional whose minimization defines $\mathbf{Y}(v)$ is constant on S, so the reverse inclusion also holds: $\mathbf{Y}(v) = S \cap \mathcal{Y}$. By (iii), $\mathcal{E} = T \cap \mathcal{Y}$. We say that $\kappa \in K$ is (*strictly*) *inward pointing* if there is $Y_0 \in \mathcal{E}$ and $Y \in \mathbf{Y}(v)$ ($Y \in \mathbf{Y}(v) \setminus \mathcal{E}$) such that κ is a positive scalar multiple of $Y - Y_0$.

(iv) $\langle\langle \kappa, D\mathbf{v}(Y_0)\kappa \rangle\rangle > 0$ for all $Y_0 \in \mathcal{E}$ and all strictly inward pointing κ.

Our main goal in this section is:

Theorem 2. *If each fixed point of F is contained in an attracting set, then the set of fixed points of F is a single attracting set.*

We explain the argument by repeatedly reducing the claim to a simpler and more technical assertion. To begin with:

Proposition 5. *If $\mathcal{E}$ is an attracting set, then there is a neighborhood $U \subseteq \mathcal{Y}$ of $\mathcal{E}$ such that the set of fixed points of F in U is $\mathcal{E}$, F has no fixed points in $\overline{U} \setminus U$, and $\Lambda(F|_{\overline{U}}) = 1$.*

Proof of Theorem 2. Proposition 5 implies that each attracting set of fixed points has a neighborhood U as above. Since (by upper semicontinuity) the set of fixed points is compact, it consists of finitely many attracting sets. Since $\Lambda(F) = 1$, and, by Additivity, $\Lambda(F)$ is equal to the number of attracting sets, there is exactly one attracting set. $\square$

We now need to prove Proposition 5. Fix an attracting set $\mathcal{E}$ and let v be the constant value of $\mathbf{v}$ on $\mathcal{E}$. Let K be the linear subspace of $\mathbb{R}^m$ that is parallel to the affine hull of $\mathbf{Y}(v)$, and let L be the linear subspace of $\mathbb{R}^m$ that is parallel to the affine hull of $\mathcal{E}$.

The general idea is to "deform" the given correspondence F to one whose entire set of fixed points is known. We fix a particular point $Y_0 \in \mathcal{E}$. Define a function $\mathbf{u} : \mathcal{Y} \times [0, 1] \to \mathbb{R}^k$ by setting

$$\mathbf{u}(Y, t) := (1 - t)\mathbf{v}(Y) + tD\mathbf{v}(Y_0)(Y - Y_0),$$

and define the correspondence $H : \mathcal{Y} \times [0, 1] \to \mathcal{Y}$ by

$$H(Y, t) := \operatorname{argmin}_{Y' \in \mathbf{Y}(v)} \langle\!\langle Y', \mathbf{u}(Y, t) \rangle\!\rangle.$$

Following notational conventions that are standard for homotopies, let H_t denote the correspondence $H(\cdot, t) : \mathcal{Y} \to \mathcal{Y}$ "at time t."

There are now three results concerning, respectively, the fixed points of H_0, H_1, and H_t for all t. The proof of the last of these is harder, and is deferred until after the proof of Proposition 5.

Lemma 4. *There is a neighborhood $U \subseteq \mathcal{Y}$ of $\mathcal{E}$ with $H_0(Y) = F(Y)$ for all $Y \in U$.*

Proof. The difference between F and H_0 is that F is defined by minimizing over all of $\mathcal{Y}$ while H_0 is defined by minimizing over $\mathbf{Y}(v)$. But by continuity (and because $\mathcal{Y}$ is a polytope) $F(Y) = \operatorname{argmin}_{\hat{Y} \in \mathcal{Y}} \langle\!\langle \hat{Y}, \mathbf{v}(Y) \rangle\!\rangle \subseteq \mathbf{Y}(v)$ for all Y in a neighborhood of $\mathcal{E}$. □

Lemma 5. *The set of fixed points of H_1 is $\mathcal{E}$.*

Proof. Fix a point $Y_0 \in \mathcal{E}$. First suppose $Y \in \mathcal{E}$. Then $Y \in \mathbf{Y}(v)$ by (ii). In addition, since $\mathbf{v}$ is constant on $\mathcal{E}$, the kernel of $D\mathbf{v}(Y_0)$ contains L, and consequently

$$Y \in H_1(Y) := \operatorname{argmin}_{Y' \in \mathbf{Y}(v)} \langle\!\langle Y', D\mathbf{v}(Y_0)(Y - Y_0) \rangle\!\rangle = \mathbf{Y}(v).$$

Thus $\mathcal{E}$ is contained in the set of fixed points of H_1.

Now suppose that $Y \in \mathcal{Y} \setminus \mathcal{E}$. We wish to show that Y is not a fixed point of H_1, so we may assume that $Y \in \mathbf{Y}(v)$ because $H_1(Y)$ is contained in this set. Then $Y - Y_0$ is strictly inward pointing, so condition (iv) implies that

$$\langle\!\langle Y, D\mathbf{v}(Y_0)(Y - Y_0) \rangle\!\rangle > \langle\!\langle Y_0, D\mathbf{v}(Y_0)(Y - Y_0) \rangle\!\rangle.$$

Since $\mathbf{u}(Y, 1) = D\mathbf{v}(Y_0)(Y - Y_0)$, it follows that $Y \notin H_1(Y)$. □

Lemma 6. *There is a neighborhood $U \subseteq \mathcal{Y}$ of $\mathcal{E}$ such that for all $0 \leqslant t \leqslant 1$ the set of fixed points of H_t in U is $\mathcal{E}$.*

Proof of Proposition 5. Let U be a neighborhood of $\mathcal{E}$ with the properties given by Lemmas 4 and 6. Any neighborhood of $\mathcal{E}$ contains a closed neighborhood, so we may assume that for all $0 \leqslant t \leqslant 1$ there are no fixed points of H_t in $\overline{U} \setminus U$. Then Continuity implies that $\Lambda(H_t|_{\overline{U}})$ is constant as a function of t, and Lemma 5 implies that $\Lambda(H_1|_{\overline{U}}) = 1$, so $\Lambda(H_0|_{\overline{U}}) = 1$. Since F agrees with H_0 on U, it follows that $\Lambda(F|_{\overline{U}}) = 1$. □

The remaining task is the proof of Lemma 6. The intuition underlying local uniqueness in coalitional bargaining is that as we change Y, say near Y_0, the agents who are included more frequently in minimal winning coalitions should have higher continuation values, which increases the expense of coalitions including many of them, in comparison with others that might be used. The net effect is to encourage change in the opposite direction, and for this reason there must be disequilibrium at points near $\mathcal{E}$ that are not actually in $\mathcal{E}$. The next result "integrates" condition (iv) in the definition of an attracting set, arriving at an algebraic expression of this idea that is valid in some neighborhood of $\mathcal{E}$.

Lemma 7. *There is a neighborhood $W \subseteq \mathbf{Y}(v)$ of $\mathcal{E}$ in the relative topology of $\mathbf{Y}(v)$ such that if $Y \in W \setminus \mathcal{E}$ and Y_1 is the point in $\mathcal{E}$ closest to Y, then*

$$\langle\!\langle Y - Y_1, \mathbf{v}(Y) - \mathbf{v}(Y_1)\rangle\!\rangle > 0.$$

Proof. Let $M := K \cap L^\perp$. Every element of K has an orthogonal decomposition as a sum of an element of L and an element of M. Since $\mathbf{Y}(v)$ is a convex polytope, there is a number $\delta > 0$ such that $\|\lambda\| \leqslant \delta\|\mu\|$ whenever $Y \in \mathbf{Y}(v) \setminus \mathcal{E}$, Y_1 is the point in $\mathcal{E}$ closest to Y, and $Y - Y_1 = \lambda + \mu$ where $\lambda \in L$ and $\mu \in M$. Let

$$C = \{(\lambda, \mu) \in L \times M\colon\ \|\lambda\| \leqslant \delta \text{ and } \|\mu\| = 1\}.$$

Then C is compact, so condition (iv) and continuity imply that there is a $\gamma > 0$ such that $\langle\!\langle \kappa, D\mathbf{v}(Y_1)\kappa\rangle\!\rangle \geqslant \gamma\|\kappa\|^2$ whenever $(\lambda, \mu) \in C$ and $\kappa = \lambda + \mu$. More generally, this inequality holds whenever $\kappa = \lambda + \mu$ with $\mu \in M$, $\lambda \in L$, and $\|\lambda\| \leqslant \delta\|\mu\|$. Choose $\varepsilon > 0$ with $\varepsilon < \gamma$, and let W be a convex neighborhood of $\mathcal{E}$ in $\mathbf{Y}(v)$ that is small enough that

$$\|D\mathbf{v}(Y) - D\mathbf{v}(Y_1)\| := \max_{\|U\|=1} \|(D\mathbf{v}(Y) - D\mathbf{v}(Y_1))U\| < \varepsilon$$

whenever $Y \in W$ and Y_1 is the point in $\mathcal{E}$ that is closest to Y.

Now fix $Y \in W \setminus \mathcal{E}$ and let Y_1 be the point in $\mathcal{E}$ closest to Y. Since

$$\mathbf{v}(Y) - \mathbf{v}(Y_1) = \int_0^1 D\mathbf{v}((1-t)Y_1 + tY)(Y - Y_1)\,dt$$

we have

$$\begin{aligned}
&\langle\!\langle Y - Y_1, \mathbf{v}(Y) - \mathbf{v}(Y_1)\rangle\!\rangle \\
&\quad = \int_0^1 \langle\!\langle Y - Y_1, D\mathbf{v}((1-t)Y_1 + tY)(Y - Y_1)\rangle\!\rangle\,dt \\
&\quad = \langle\!\langle Y - Y_1, D\mathbf{v}(Y_1)(Y - Y_1)\rangle\!\rangle \\
&\qquad + \int_0^1 \langle\!\langle Y - Y_1, (D\mathbf{v}((1-t)Y_1 + tY) - D\mathbf{v}(Y_1))(Y - Y_1)\rangle\!\rangle\,dt \\
&\quad \geqslant \gamma\|Y - Y_1\|^2 - \int_0^1 \|Y - Y_1\| \cdot \|(D\mathbf{v}((1-t)Y_1 + tY) - D\mathbf{v}(Y_1))(Y - Y_1)\|\,dt \\
&\quad \geqslant (\gamma - \varepsilon)\|Y - Y_1\|^2 > 0. \qquad \square
\end{aligned}$$

We now combine the last result with the definition of H_t.

Proof of Lemma 6. Let $W \subseteq \mathbf{Y}(v)$ be as in Lemma 7. Let $U \subseteq \mathcal{Y}$ be a neighborhood of $\mathcal{E}$ such that $U \cap \mathbf{Y}(v) \subseteq W$. Fix a particular $t \in [0, 1]$. We already know, from Lemma 5, that the set of fixed points of H_1 is $\mathcal{E}$, so we may assume that $t < 1$.

First suppose that $Y \in \mathcal{E}$. Then $D\mathbf{v}(Y_0)(Y - Y_0) = 0$ because $\mathbf{v}$ is constant on $\mathcal{E}$. Thus $\mathbf{u}(Y, t) = (1 - t)\mathbf{v}(Y) = (1 - t)v$, and Y is an element of $H_t(Y) = \mathbf{Y}(v)$. Thus $\mathcal{E}$ is contained in the set of fixed points of H_t.

We need to show that $U \setminus \mathcal{E}$ does not contain any fixed points of H_t. Fix $Y \in U \setminus \mathcal{E}$. The image of H_t is contained in $\mathbf{Y}(v)$, so we may assume that $Y \in \mathbf{Y}(v)$. Let Y' be the point in $\mathcal{E}$ closest to Y. Then $\mathbf{v}(Y') = v$, and $Y - Y' \in K$ so $\langle\langle Y - Y', \mathbf{v}(Y')\rangle\rangle = 0$. Therefore Lemma 7 implies that

$$\big\langle\!\big\langle Y - Y', \mathbf{v}(Y)\big\rangle\!\big\rangle = \big\langle\!\big\langle Y - Y', \mathbf{v}(Y) - \mathbf{v}\big(Y'\big)\big\rangle\!\big\rangle > 0.$$

We have $D\mathbf{v}(Y_0)(Y_0 - Y') = 0$, as explained above, so the definition of an attracting set gives

$$\big\langle\!\big\langle Y - Y', D\mathbf{v}(Y_0)(Y - Y_0)\big\rangle\!\big\rangle = \big\langle\!\big\langle Y - Y', D\mathbf{v}(Y_0)\big(Y - Y'\big)\big\rangle\!\big\rangle > 0.$$

Multiplying the last two inequalities by $1 - t$ and t, then summing, gives

$$\big\langle\!\big\langle Y - Y', \mathbf{u}(Y, t)\big\rangle\!\big\rangle > 0.$$

That is, $\langle\langle Y', \mathbf{u}(Y, t)\rangle\rangle < \langle\langle Y, \mathbf{u}(Y, t)\rangle\rangle$, and consequently $Y \notin H_t(Y)$. □

6. The proof of Theorem 1

We now return to the framework of Section 4, so $\mathbf{v}(Y) = A(Y)^{-1}p$. For $Y \in \mathcal{Y}$ and $v \in \mathbb{R}^n$ let

$$\langle Y, v\rangle = p^T Y^T \Delta v,$$

and for $v \in \mathbb{R}^n$ let $\mathbf{Y}(v) = \operatorname{argmin}_{Y \in \mathcal{Y}} \langle Y, v\rangle$. As before $F : \mathcal{Y} \to \mathcal{Y}$ is the correspondence $F(Y) = \mathbf{Y}(\mathbf{v}(Y))$. Fix a particular fixed point Y_0 of F, and let:

$$v := \mathbf{v}(Y_0), \qquad m := \mathbf{m}(Y_0), \qquad \mathcal{E} := \mathbf{m}^{-1}(m).$$

Lemma 1 implies that for all $Y \in \mathcal{E}$, (v, Y) is a reduced equilibrium, so $\mathbf{v}$ is constant on $\mathcal{E}$, and $\mathcal{E} \subseteq \mathbf{Y}(v)$. Since $\mathbf{m}$ is a linear function, $\mathcal{E}$ is the intersection of $\mathcal{Y}$ with the affine hull of $\mathcal{E}$. Thus conditions (i)–(iii) of the definition of an attracting set are satisfied. We will show that (iv) also holds, so that $\mathcal{E}$ is an attracting set for $\mathbf{Y}$ and $\mathbf{v}$, and since Y_0 is an arbitrary fixed point of F, Theorem 2 will then imply that it is the entire set of fixed points of F. As we pointed out earlier, the other assertions of Theorem 1 follow from this, so that result will be established.

The remainder of this section is devoted to the proof that (iv) also holds. Let K be the linear subspace of the Euclidean space containing $\mathcal{Y}$ that is parallel to the affine hull of $\mathbf{Y}(v)$, and let L be the linear subspace that is parallel to the affine hull of $\mathcal{E}$. Condition (iv) follows from Lemma 8 below.

Lemma 8. *If $\kappa \in K$ is strictly inward pointing, then*

$$\big\langle \kappa, D\mathbf{v}(Y_0)\kappa\big\rangle = p^T \kappa^T \Delta D\mathbf{v}(Y_0)\kappa > 0.$$

The rest of the section is devoted to the proof of this. We begin with two technical facts.

Lemma 9. *If $\kappa \in K$ is inward pointing and $\kappa p = 0$, then κ is not strictly inward pointing.*

Proof. We know that κ is a positive scalar multiple of $Y - \tilde{Y}_0$ for some $Y \in \mathbf{Y}(v)$ and $\tilde{Y}_0 \in \mathcal{E}$. We have

$$\mathbf{m}(\tilde{Y}_0 + t\kappa) = p_0\mathbf{e} + (\tilde{Y}_0 + t\kappa)p = p_0\mathbf{e} + \tilde{Y}_0 p = \mathbf{m}(\tilde{Y}_0)$$

for all t (where $\mathbf{e} = (1, \dots, 1) \in \mathbb{R}^n$) so $\tilde{Y}_0 + t\kappa \in \mathcal{E}$ for all $t \geqslant 0$ such that $\tilde{Y}_0 + t\kappa \in \mathbf{Y}(v)$. In particular, $Y \in \mathcal{E}$. □

Lemma 10. *For all $\kappa \in K$, $\kappa^T \Delta v = 0$.*

Proof. Observe that κ is a scalar multiple of a difference $Y' - Y''$ of two vectors in $\mathbf{Y}(v)$, and that these are both minimizers of $p^T Y^T \Delta v$. Since the columns of Y can be chosen independently, it follows that Y'' and Y' both minimize each component of $Y^T \Delta v$. □

Let $V := \nabla(v)$.

Lemma 11. *For all $\kappa \in K$, $D\mathbf{v}(Y_0)\kappa = A^{-1} V \Delta\kappa p$.*

Proof. Differentiating the equation $A(Y)\mathbf{v}(Y) = p$ gives

$$A D\mathbf{v}(Y_0)\kappa + \big(DA(Y_0)\kappa\big)v = 0,$$

so the assertion is equivalent to $-(DA(Y_0)\kappa)v = V\Delta\kappa p$. Evaluating the derivative of A shows that this is equivalent to

$$\big(\nabla(\kappa p) - P\kappa^T\big)\Delta v = V\Delta\kappa p.$$

Since $\kappa^T \Delta v = 0$ this is true if and only if $\nabla(\kappa p)\Delta v = V\Delta\kappa p$. But commutativity of multiplication of diagonal matrices, and the fact that $\nabla(x)y = \nabla(y)x$ for all $x, y \in \mathbb{R}^n$, gives

$$\nabla(\kappa p)\Delta v = \Delta\nabla(\kappa p)v = \Delta V\kappa p = V\Delta\kappa p. \qquad \square$$

Our general approach is inspired by Theorem 3 of Debreu [27], which asserts that if C is an $n \times n$ matrix, B is an $m \times n$ matrix, and $x^T C x > 0$ for all nonzero x such that $Bx = 0$, then there is some $\lambda > 0$ such that $C + \lambda B^T B$ is positive definite.

Fix a strictly inward pointing $\kappa \in K$. We premultiply the conclusion of the last result by $p^T\kappa^T$, then put the resulting quantity in a symmetric form:

$$\begin{aligned}
p^T\kappa^T\Delta D\mathbf{v}(Y_0)\kappa &= p^T\kappa^T\Delta A^{-1}V\Delta\kappa p \\
&= \frac{1}{2}p^T\kappa^T\Delta A^{-1}V\Delta\kappa p + \frac{1}{2}\big(p^T\kappa^T\Delta A^{-1}V\Delta\kappa p\big)^T \\
&= \frac{1}{2}p^T\kappa^T\Delta\big(A^{-1}V + V^T\big(A^{-1}\big)^T\big)\Delta\kappa p \\
&= \frac{1}{2}p^T\kappa^T\Delta A^{-1}\big(VA^T + AV^T\big)\big(A^{-1}\big)^T\Delta\kappa p.
\end{aligned}$$

Since $A^{-1}p = v$ and $\kappa^T\Delta v = 0$, we have

$$p^T\kappa^T\Delta A^{-1}p = p^T\kappa^T\Delta v = 0.$$

Therefore, for any $\gamma \in \mathbb{R}$,

$$p^T\kappa^T\Delta D\mathbf{v}(Y_0)\kappa = \frac{1}{2}p^T\kappa^T\Delta A^{-1}G(\gamma)\big(A^{-1}\big)^T\Delta\kappa p$$

where

$$G(\gamma) := VA^T + AV^T - \gamma p v^T - \gamma v p^T.$$

Since $\kappa p \neq 0$ (by Lemma 9) and A^{-1} and Δ are nonsingular, to prove Lemma 8 it suffices to find γ such that $G(\gamma)$ is positive definite.

The final step in the proof is:

Lemma 12. *If* $\max_i \delta_i \leqslant \gamma < 1$*, then* $G(\gamma)$ *is a symmetric nonsingular* M*-matrix, so it is positive definite.*

Proof. Let $G(\gamma) = (g_{ij})$. Fix an $i = 1, \dots, n$. We have $a_{ii} = 1 - \delta_i(m_i - p_i)$ and

$$g_{ii} = 2a_{ii}v_i - 2\gamma p_i v_i = 2\big(1 - \delta_i(m_i - p_i)\big)v_i - 2\gamma p_i v_i. \qquad (*)$$

Recall that

$$m_i = p_0 + \sum_{j=1}^n p_j y_{ij} \leqslant p_0 + \sum_{j=1}^n p_j = 1,$$

so

$$g_{ii} \geqslant 2v_i\big(1 - \delta_i(1 - p_i) - \gamma p_i\big),$$

so $g_{ii} > 0$ because $\delta_i, \gamma < 1$.

When $i \neq j$ we have $a_{ij} = p_i \delta_j y_{ji}$, so that

$$g_{ij} = a_{ij}v_j + a_{ji}v_i - \gamma(p_i v_j + p_j v_i) = (\delta_j y_{ji} - \gamma)p_i v_j + (\delta_i y_{ij} - \gamma)p_j v_i. \qquad (**)$$

Therefore $g_{ij} \leqslant 0$ when $i \neq j$ because $\gamma \geqslant \delta_i$ for all i.

The remaining condition that we need to verify is that $G(\gamma)$ is dominant diagonal. Summing equations $(*)$ and $(**)$ above, then recognizing that $y_{ii} = 1$, yields

$$\sum_{j=1}^n g_{ij} = C - \gamma\left(p_i \sum_{j=1}^n v_j + v_i \sum_{j=1}^n p_j\right)$$

where

$$C = 2(1 - \delta_i m_i)v_i + p_i \sum_{j=1}^n v_j \delta_j y_{ji} + \delta_i v_i \sum_{j=1}^n p_j y_{ij}.$$

Substituting $m_i = p_0 + \sum_{j=1}^n p_j y_{ij}$ yields

$$C = 2(1 - \delta_i p_0)v_i + p_i \sum_{j=1}^n v_j \delta_j y_{ji} - \delta_i v_i \sum_{j=1}^n p_j y_{ij}.$$

Condition (a) of the definition of reduced equilibrium gives

$$p_i \sum_{j=1}^n v_j \delta_j y_{ji} = \delta_i v_i \left(\sum_{j=1}^n p_j y_{ij}\right) + (p_0 \delta_i - 1)v_i + p_i,$$

so $C = (1 - \delta_i p_0)v_i + p_i$, and consequently

$$\sum_{j=1}^{n} g_{ij} = (1 - \delta_i p_0)v_i + p_i - \gamma\left(p_i \sum_{j=1}^{n} v_j + v_i \sum_{j=1}^{n} p_j\right).$$

Since $\sum_{j=1}^{n} p_j = 1 - p_0$ we have

$$\sum_{j=1}^{n} g_{ij} = \bigl(1 - \delta_i p_0 + \gamma(1 - p_0)\bigr)v_i + p_i\left(1 - \gamma \sum_{j=1}^{n} v_j\right).$$

In order to show that $\sum_{j=1}^{n} g_{ij}$ is positive it now suffices to show that $\sum_{j=1}^{n} v_j \leqslant 1$. Summing condition (a) of the definition of reduced equilibrium over i, simplifying, and substituting $\sum_{i=1}^{n} p_i = 1 - p_0$, leads to

$$\sum_{i=1}^{n} v_i = p_0 \sum_{i=1}^{n} \delta_i v_i + \sum_{i=1}^{n} p_i \leqslant p_0 \sum_{i=1}^{n} v_i + 1 - p_0.$$

Therefore $(1 - p_0)\sum_{i=1}^{n} v_i \leqslant 1 - p_0$, which implies the desired conclusion. □

7. Future research

Many questions and issues remain unresolved. As mentioned earlier, Kalandrakis [51] provides an algorithm for computing the vector v of stationary subgame perfect equilibrium expected payoffs for the Eraslan [34] model. Such an algorithm for the more general model of this paper would greatly expand the range of theoretical and empirical work that can be supported computationally. The definition of reduced equilibrium has a combinatoric aspect, namely which coalitions are least cost for each proposer, and a numerical aspect. Unlike linear programming or Nash equilibrium for two person games, even once the combinatoric aspect has been solved, the numerical aspect is nonlinear, so there is little hope of finding an algorithm based on linear pivoting such as the simplex algorithm for linear programming, the Lemke–Howson algorithm (Lemke and Howson [57]) for two player games, or the Lemke [56] algorithm for linear complementary problems. Another potential approach to computation is based on homotopy: starting with recognition probabilities for which the stationary subgame perfect equilibrium payoff vector is known, follow a path of (recognition probability vector, payoff vector) pairs until one reaches the recognition probabilities of interest. Due to the nonlinearity, this approach is likely to be difficult to analyze theoretically, but may well be practical in many applications.

This problem is interesting from the point of view of computer science, which has an active line of literature concerned with the complexity of computing fixed points, with special emphasis on the computation of Nash equilibria. (See Etessami and Yannakis [36] and Papadimitriou [79].) Computation of the vector of continuation values is a special problem in this class because it is nonlinear and a unique solution is guaranteed, but our proof of uniqueness does not supply an algorithm. Recall that **P** is the class of decision problems that can be decided by a Turing machine whose running time is bounded by a polynomial function of the size of the input, and **NP** is the class of decision problems such that a positive answer has a “witness” that can be verified in polynomial time. For example, if a graph has a clique of size k (that is, k vertices such that any two are the endpoints of an edge) then such a clique is a witness for that fact. Similarly, **coNP** is the class of decision problems for which a negative answer has an easily verified witness; obviously it is the class of decision problems whose negations are in **NP**. It is not known whether the intersection of **NP** and **coNP** contains problems that are not in **P** (whether

NP contains problems that are not in **P** is currently one of the most important open problems in mathematics) but there are currently very few problems in the intersection of **NP** and **coNP** for which no polynomial time algorithm is known. The best known examples are related to objects whose unique existence is guaranteed, specifically factoring of integers and equilibrium payoffs of "simple" zero sum stochastic games (cf. Johnson [46]). Decision problems related to v are perhaps a source of such problems, but because the components of v need not be rational, it is not clear that such problems can be placed in **NP**.

There are many interesting questions concerning the vector v, which can be investigated either theoretically or computationally. Eraslan [34] shows that in the case of k-majority rule, if the recognition probabilities are all the same, then the costs of players, as potential members of a coalition, are ordered in the same way as the discount factors. That is, if $\delta_i \leqslant \delta_j$, then $\delta_i v_i \leqslant \delta_j v_j$. She also shows that if all players have the same discount factor, then the costs of players are ordered in the same way as the recognition probabilities, i.e., $p_i \leqslant p_j$, then $\delta_i v_i \leqslant \delta_j v_j$. It is natural to ask whether these results generalize to coalition structures that are symmetric in the sense that for any i and j there is a permutation of N that maps i to j and preserves (in the natural sense) the structure of the sets of minimal winning coalitions. There are a host of additional questions concerning monotonicity of v_i as an agent i becomes more or less powerful due to changes in the structure of the sets of minimal winning coalitions.

Our results permit the definition of a new power index. The Shapley value (Shapley [87], Shapley and Shubik [88]) is a function that assigns a vector of payoffs to each TU game, and the Shapley–Shubik (Shapley and Shubik [88]) power index is the application of the Shapley value to simple games. The power indices of Banzhaf [7,8] Deegan and Packel [28] and Johnston [47] are functions with the same domain and range: each assigns a vector of individual "powers" to each simple game. In our framework there is a simple game (the system of winning coalitions) and other parameters, namely the recognition probabilities and the discount factors. To obtain a power index comparable to those mentioned in Section 2 one may take the limit of our vector of equilibrium continuation payoffs, for the case of symmetric recognition probabilities, as the common discount factor goes to one. The power index obtained in this way has clear noncooperative foundations, in line with the Nash program. These are certainly open to question in some applications, and can be compared with other noncooperative foundations for cooperative solutions (e.g., Gul [39]).

Many interesting topics concern generalizations or variations of the model. It would be desirable to extend this paper's model to allow for different proposer-coalition pairs to generate pies of different size. Among other things, one could then investigate the hypothesis that the coalitions that form are the most productive. Whether our uniqueness result extends to such a model is an important open question. There seems to be considerable scope for additional work on alternative bargaining protocols such as Chatterjee et al. [22], Ray [85], and Kawamori [52]. Another possibility studied by Montero [69] is to search for vectors of recognition probabilities that are *self-confirming* in the sense that they coincide with the resulting vector of continuation values. As we mentioned in the introduction, she shows that points in the nucleus of a proper simple game have this property. Comparison of the properties of the various power indices seems like an interesting direction for theoretical investigation.

Appendix A

This appendix gives a precise description of the bargaining game and shows that the expected payoff vectors generated by its stationary subgame perfect equilibria satisfy the characterization given by the definition of reduced equilibrium. This involves a certain amount of advanced measure theory, but, taking these tools as given, the work is straightforward and unsurprising. Since, quite understandably, many papers in this area do not fill in these details, we hope this appendix may be useful as a reference or model for other authors.

We describe the progress of the game within a single active period in terms of the space of *within-period histories*

$$H := H^0 \cup H^1 \cup \cdots \cup H^n.$$

Here H^0 is the set Σ_n of permutations of $\{1, \ldots, n\}$. There is a given probability distribution q on Σ_n. If $\sigma \in \Sigma_n$ is realized, then $\sigma(1)$ is the proposer, and the other agents vote on her proposal sequentially in the order $\sigma(2), \ldots, \sigma(n)$. Therefore $p_i/(1 - p_0) = \sum_{\sigma(1)=i} q(\sigma)$ for all i. (One consequence of our analysis here is that other aspects of q have no influence on equilibrium outcomes.) After σ is realized, $\sigma(1)$ chooses a proposal $x \in \Pi$, so $H^1 = \Sigma_n \times \Pi$. The agents then vote in order, with each seeing σ, the proposal, and the earlier agents' votes, then choosing an element of {Yes, No}, so that for each $k = 1, \ldots, n$,

$$H^k := \Sigma_n \times \Pi \times \{\text{Yes}, \text{No}\}^{k-1}.$$

For $k = 0, \ldots, n - 1$ and $i = 1, \ldots, n$ let H_i^k be the set of within-period histories in H^k at which agent i chooses. Thus $H_i^0 = \{\sigma \in H^0\colon \sigma(1) = i\}$, and for $k = 1, \ldots, n - 1$ we have

$$H_i^k = \big\{(\sigma, x, b_{\sigma(2)}, \ldots, b_{\sigma(k)}) \in H^k\colon \sigma(k + 1) = i\big\}.$$

The measure-theoretic description of behavior strategies involves transition probabilities. In general, if $(\Omega, \mathcal{A})$ is a measurable space, let $\mathcal{P}(\Omega)$ be the set of probability measures on Ω. If $(\Omega_1, \mathcal{A}_1)$ and $(\Omega_2, \mathcal{A}_2)$ are measurable spaces, a function $P_{12} : \Omega_1 \to \mathcal{P}(\Omega_2)$ is a *transition probability* if

$$P_{12}(\cdot)(A_2) : \Omega_1 \to [0, 1]$$

is measurable for all $A_2 \in \mathcal{A}_2$. Let $\mathcal{P}(\Omega_1, \Omega_2)$ be the set of transition probabilities from Ω_1 to Ω_2.

A (stationary or Markov) *proposer strategy* for agent $i = 1, \ldots, n$ is a transition probability

$$\pi_i \in \mathcal{P}\big(H_i^0, \Pi\big).$$

Fix proposer strategies $\pi_1, \ldots, \pi_n$ for the various agents. A *responder strategy* for agent i is a transition probability

$$\rho_i \in \mathcal{P}\big(H_i^1 \cup \cdots \cup H_i^{n-1}, \{\text{Yes}, \text{No}\}\big).$$

Fix responder strategies $\rho_1, \ldots, \rho_n$. Let

$$\pi \in \mathcal{P}\big(H^0, \Pi\big)$$

be the transition probability that agrees with π_i on each H_i^0. Abusing notation, we also use the symbol π to denote the profile $(\pi_1, \ldots, \pi_n)$; the correct interpretation will always be clear from context. Let $\rho := (\rho_1, \ldots, \rho_n)$, and for $k = 1, \ldots, n - 1$ let

$$\rho^k \in \mathcal{P}\big(H^k, \{\text{Yes}, \text{No}\}\big)$$

be the transition probability that agrees with ρ_i on each H_i^k.

We now need to define and characterize the measures on $H^0, \dots, H^n$ induced by (π, ρ). The following result generalizes Fubini's theorem and is a fundamental result for the theory of Markov chains.

Lemma A.1. *For any $\lambda \in \mathcal{P}(\Omega_1)$ and $P_{12} \in \mathcal{P}(\Omega_1, \Omega_2)$ there is a unique probability measure $\lambda \otimes P_{12} \in \mathcal{P}(\Omega_1 \times \Omega_2)$ satisfying*

$$(\lambda \otimes P_{12})(A_1 \times A_2) = \int\limits_{A_1} P_{12}(\cdot)(A_2)\, d\lambda$$

for all $A_1 \in \mathcal{A}_1$ and $A_2 \in \mathcal{A}_2$. If $X : \Omega_1 \times \Omega_2 \to \mathbb{R}_+$ is integrable, then

$$\int\limits_{\Omega_2} X(\omega_1, \omega_2) P_{12}(\omega_1)(d\omega_2)$$

is a measurable function of ω_1, and

$$\int\limits_{\Omega_1 \times \Omega_2} X d(\lambda \otimes P_{12}) = \int\limits_{\Omega_1} \Bigg[\int\limits_{\Omega_2} X(\omega_1, \omega_2) P_{12}(\omega_1)(d\omega_2) \Bigg] \lambda(d\omega_1).$$

Proof. E.g., Proposition III.2.1 of Neveu [72]. □

The measure on H^n induced by q and (π, ρ) is now seen to be

$$q \otimes \pi \otimes \rho^1 \otimes \dots \otimes \rho^{n-1}.$$

Let $\mathcal{O}$ be the disjoint union of Π and {Fail}, endowed with the obvious σ-algebra, and define the *outcome function* $o : H^n \to \mathcal{O}$ by setting

$$o(\sigma, x, b_{\sigma(2)}, \dots, b_{\sigma(n)}) = \begin{cases} x, & \{\sigma(1)\} \cup \{j\colon b_j = \text{Yes}\} \in \mathcal{S}_{\sigma(1)}, \\ \text{Fail}, & \text{otherwise.} \end{cases}$$

The measure on $\mathcal{O}$ induced by (π, ρ) is

$$\big(q \otimes \pi \otimes \rho^1 \otimes \dots \otimes \rho^{n-1}\big) \circ o^{-1}.$$

In order to define subgame perfection we need to also define the measures on $\mathcal{O}$ induced by beginning with a particular partial history and continuing according to (π, ρ). For $h \in H$ we define

$$\kappa(h, \pi, \rho) := \begin{cases} (\delta_h \otimes \pi \otimes \rho^1 \otimes \dots \otimes \rho^{n-1}) \circ o^{-1}, & h \in H^0, \\ (\delta_h \otimes \rho^j \otimes \dots \otimes \rho^{n-1}) \circ o^{-1}, & h \in H^j,\ j = 1, \dots, n-1, \\ \delta_h \circ o^{-1}, & h \in H^n. \end{cases}$$

Here δ_h is the (Dirac) probability measure on H that assigns all probability to h. (This symbol does not appear below, so confusion with the discount factors will not be possible.) For $h \in H$ and $v \in V$ we define $\tau(h, \pi, \rho, v) \in [0, 1]^n$ by setting

$$\tau_i(h, \pi, \rho, v) = \kappa(h, \pi, \rho)\big(\{\text{Fail}\}\big) \cdot \delta_i v_i + \int\limits_{\Pi} x_i \kappa(h, \pi, \rho)(dx).$$

We say that (π, ρ) is a *within period subgame perfect v-equilibrium* if

$$v_i = p_0 \cdot \delta_i v_i + (1 - p_0) \sum_{\sigma \in \Sigma_n} q(\sigma)\tau_i(\sigma, \pi, \rho, v) \qquad (\dagger)$$

for all $i = 1, \ldots, n$ and

$$\tau_i(h, \pi, \rho, v) \geqslant \tau_i\big(h, (\pi_1, \ldots, \tilde{\pi}_i, \ldots, \pi_n), (\rho_1, \ldots, \tilde{\rho}_i, \ldots, \rho_n), v\big) \qquad (\ddagger)$$

for all $h \in H$, all $i = 1, \ldots, n$, and all proposer strategies $\tilde{\pi}_i$ and responder strategies $\tilde{\rho}_i$ for i.

For $k = 1, \ldots, n$ and $h = (\sigma, x, b_{\sigma(2)}, \ldots, b_{\sigma(k-1)}) \in H^k$ let

$$C(h) := \big\{\sigma(1)\big\} \cup \big\{\sigma(j)\colon\ 2 \leqslant j \leqslant k-1 \text{ and } b_{\sigma(j)} = \text{Yes}\big\}$$

be the set of agents who have voted to approve x. For k and h as above and $v \in \mathcal{V}$ let

$$D(h, v) := \big\{\sigma(j)\colon\ j = k, \ldots, n \text{ and } x_{\sigma(j)} > \delta_{\sigma(j)} v_{\sigma(j)}\big\}$$

be the set of agents in $\{\sigma(k), \ldots, \sigma(n)\}$ who prefer x to their disagreement payoff, and let

$$\overline{D}(h, v) := \big\{\sigma(j)\colon\ j = k, \ldots, n \text{ and } x_{\sigma(j)} \geqslant \delta_{\sigma(j)} v_{\sigma(j)}\big\}$$

be the set of agents in $\{\sigma(k), \ldots, \sigma(n)\}$ who do not prefer their disagreement payoff to x.

Lemma A.2. *If (π, ρ) is a within period subgame perfect v-equilibrium and $C(h) \cup \overline{D}(h, v) \notin \mathcal{S}_{\sigma(1)}$, then the probability (conditional on h) of implementing the proposal x is zero.*

Proof. The claim is certainly correct if $k = n$, since then the last agent will vote according to her interest if her vote makes a difference. Suppose it is true with k replaced with $k + 1$. If $\sigma(k) \in \overline{D}(h, v)$, then regardless of how $\sigma(k)$ votes the proposal will certainly not be implemented, and if $\sigma(k) \notin \overline{D}(h, v)$, then $\sigma(k)$ can insure rejection by voting against, and will only vote in favor if that also results in rejection with probability one. □

We now study the relationship between within period subgame perfect v-equilibria and reduced equilibrium. From this point forward we assume that supersets of winning coalitions are winning. That is, for each i and $S_i \in \mathcal{S}_i$, if $S_i \subseteq T_i \subseteq N$, then $T_i \in \mathcal{S}_i$. We begin by showing that if v is the first component of a reduced equilibrium, then there is a within period subgame perfect v-equilibrium. For $i = 1, \ldots, n$ define

$$\mathbf{w}_i(v) = 1 - \min_{S \in \mathcal{S}_i} \sum_{j \in S} \delta_j v_j.$$

For each i let

$$\mathcal{S}_i^*(v) := \operatorname{argmin}_{S \in \mathcal{S}_i} \sum_{j \in S} \delta_j v_j$$

be the set of minimum cost coalitions.

Proposition A.3. *If (v, Y) is a reduced equilibrium, then there is (π, ρ) such that:*

(i) *for each $\sigma \in \Sigma_n$, $\pi_{\sigma(1)}$ assigns all probability to proposals such that, for some $S \in \mathcal{S}^*_{\sigma(1)}(v)$, each member $j \neq \sigma(1)$ of S receives $\delta_j v_j$, $\sigma(1)$ receives $\delta_{\sigma(1)} v_{\sigma(1)} + 1 - \sum_{j \in S} \delta_j v_j$, and all other agents receive 0;*

(ii) *for all distinct* $i, j = 1, \dots, n$, *the proposer strategy* π_i *assigns probability* y_{ij} *to proposals in which* j *receives* $\delta_j v_j$;

(iii) *for each* i, *in response to a proposal* x *with* $x_i \geqslant \delta_i v_i$, ρ_i *assigns all probability to voting to accept, and in response to a proposal* x *with* $x_i < \delta_i v_i$, ρ_i *assigns all probability to voting to reject.*

Such a (π, ρ) *is a within period subgame perfect* v*-equilibrium.*

Proof. Since (v, Y) is a reduced equilibrium, each y_i minimizes expenditure on coalition partners and is achievable as a convex combination of vectors associated with winning coalitions for i. Therefore there exists π satisfying (i) and (ii) and ρ satisfying (iii).

We now verify that (π, ρ) is a within period subgame perfect v-equilibrium. Conditions (i) and (ii) imply that

$$\tau_i(\sigma, \pi, \rho, v) = \begin{cases} 1 - \min_{S \in \mathcal{S}_i} \sum_{j \in S \setminus \{i\}} \delta_j v_j, & \sigma(1) = i, \\ y_{\sigma(1)i} \delta_i v_i, & \sigma(1) \neq i. \end{cases}$$

From this and the assumption that (v, Y) is a reduced equilibrium it follows that

$$p_0 \cdot \delta_i v_i + (1 - p_0) \sum_{\sigma \in \Sigma_n} q(\sigma) \tau_i(\sigma, \pi, \rho, v) = \mathbf{m}_i(Y) \delta_i v_i + p_i \mathbf{w}_i(v) = v_i.$$

That is, (†) holds.

Since, under ρ, a responder's vote has no effect on the votes of subsequent voters, it is clear there is no improving deviation from ρ, either for a voter who is content with the proposal, or for one who is not because voting in favor of a proposal cannot sabotage it. Given that responders are playing ρ, the proposer $\sigma(1)$ achieves the expected payoff $\delta_{\sigma(1)} v_{\sigma(1)} + \mathbf{w}_{\sigma(1)}(v)$, and Lemma A.2 implies that there is no way to do better than this. Thus (‡) holds. □

For the remainder we fix a within period subgame perfect v-equilibrium (π, ρ). Our goal is to show that there is an associated reduced equilibrium that has v as its first component.

Lemma A.4. *If* $C(h) \cup D(h, v) \in \mathcal{S}_{\sigma(1)}$, *then, conditional on* h, *the proposal* x *will be implemented with probability one.*

Proof. The claim is certainly correct if $k = n$, since then the last agent will vote according to her interest if her vote makes a difference. Suppose it is true with k replaced with $k + 1$. If $\sigma(k) \notin D(h, v)$, then passage of the proposal is certain, by virtue of the induction hypothesis, regardless of how $\sigma(k)$ votes. (In particular, $\sigma(k)$ cannot defeat the proposal by voting in favor of it.) If $\sigma(k) \in D(h, v)$, then $\sigma(k)$ can insure passage of the proposal by voting in favor, and in a subgame perfect equilibrium will not vote against unless passage is also guaranteed in that event. □

For each $\sigma \in \Sigma_n$ and $S \in \mathcal{S}_{\sigma(1)}$ let $\eta^*_{\sigma(1)}(S)$ be the probability that $\pi_{\sigma(1)}(A, \sigma)$ assigns to the allocation in which each member j of S other than $\sigma(1)$ receives $\delta_j v_j$, $\sigma(1)$ receives $\delta_{\sigma(1)} v_{\sigma(1)} + 1 - \sum_{j \in S} \delta_j v_j$, and all other agents receive 0.

Lemma A.5. *For each* σ

$$\sum_{S \in \mathcal{S}^*_{\sigma(1)}(v)} \eta^*_\sigma(S) = 1,$$

and the proposal is accepted with probability one.

Proof. Lemma A.4 implies that $\sigma(1)$ can insure the implementation of any proposal x such that $\{j \neq \sigma(1)\colon x_j > \delta_j v_j\} \in \mathcal{S}_{\sigma(1)}$. Lemma A.2 implies that $\sigma(1)$ cannot hope to implement a proposal x with $\{j \neq \sigma(1)\colon x_j \geqslant \delta_j v_j\} \notin \mathcal{S}_{\sigma(1)}$. Combining these two results, we find that $\sigma(1)$ has expected utility $\delta_{\sigma(1)} v_{\sigma(1)} + \mathbf{w}_{\sigma(1)}(v)$, and that this expected utility can be achieved only if a proposal is implemented with probability one. □

For each i and j let

$$y_{ij} = \sum_{\sigma \in \Sigma_n, \sigma(1)=i} q(\sigma) \sum_{S \in \mathcal{S}_i} \eta^*_i(S),$$

and let Y be the matrix with these components.

Proposition A.6. (v, Y) *is a reduced equilibrium.*

Proof. By construction $y_i \in \operatorname{argmin}_{y'_i \in \mathcal{Y}_i} \sum_j y'_{ij} \delta_j v_j$. Since (π, ρ) is a within period subgame perfect v-equilibrium we have

$$v_i = p_0 \cdot \delta_i v_i + (1 - p_0) \sum_{\sigma \in \Sigma_n} q(\sigma) \tau_i(\sigma, \pi, \rho, v) = \mathbf{m}_i(Y) \delta_i v_i + p_i \mathbf{w}_i(v). \qquad \square$$

Ideally we should also demonstrate that a within period subgame perfect v-equilibrium is also a stationary subgame perfect equilibrium in the sense that, after any history of the larger game, playing according to the equilibrium strategies is preferable to any deviation, includings deviations to strategies that are not stationary. A formal definition of stationary subgame perfect equilibrium would require a description of the space of all strategies, including those that are not stationary, which would be tedious and serve little purpose. Instead we briefly indicate how the relevant issues can be handled.

Suppose that (π, ρ) is a within period subgame perfect v-equilibrium. For any agent i, once stationary strategies for the other agents are fixed, the game becomes a stationary discounted dynamic program. For such problems it is well known that if, for every state, there is an optimal strategy, then each state has a value. In addition, a strategy is optimal if and only if it is myopically optimal, in the sense of maximizing the expectation of the sum of the current reward and the discounted value of next period's state. In particular, there is a stationary strategy that is optimal for every state. Thus (π, ρ) is a stationary subgame perfect equilibrium because it correctly values failure to reach agreement in the current period and if, after some history, there was an improving deviation, there would necessarily be a violation of (‡).

Conversely, suppose that (π, ρ) is a stationary subgame perfect equilibrium. Then each history has an associated vector of values, which depends only on the current state and not on the prior history of play. In particular there is a vector v of expected payoffs conditional on play entering a given period, and (‡) holds because this same vector of expected values characterizes expected payoffs if no agreement is reached in this period. That is, there is a v such that (π, ρ) is a within period subgame perfect v-equilibrium.

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