

Special thanks to Martin Osborne for his careful reading. See the corrections on pages 31, 36, 38 & 39.

Uniqueness of stationary equilibrium payoffs in the Baron–Ferejohn model with risk-averse players

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I study a multilateral sequential bargaining model among risk-averse players in which the players may differ in their probability of being selected as the proposer and the rate at which they discount future payoffs. For games in which agreement requires less than unanimous consent, I characterize the set of stationary subgame perfect equilibrium payoffs. With this characterization, I establish the uniqueness of the equilibrium payoffs. For the case where the players have the same discount factor, I show that the payoff to a player is non-decreasing in his probability of being selected as the proposer. For the case where the players have the same probability of being selected as the proposer, I show that the payoff to a player is non-decreasing in his discount factor. This generalizes earlier work by allowing the players to be risk averse.

Key words non-cooperative bargaining, multilateral bargaining, legislative bargaining, uniqueness, risk-averse players

JEL classification C72, C78, D70

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1 Introduction

In their influential work, Baron and Ferejohn (1989) present legislative bargaining game with risk-neutral players.¹ In each period, one of the players is randomly selected to make a proposal as to how to divide a fixed pie, and agreement requires the consent of a simple majority, otherwise the process is repeated until agreement is reached with payoffs discounted geometrically at a common rate for all players. Baron and Ferejohn (1989) show that any division of the pie can be supported as a subgame perfect equilibrium if there are at least five players and the players are sufficiently patient. In light of this result, they restrict attention to stationary strategies. While in their model they allow for the probabilities with which players are selected to be the proposer to differ, they only establish the uniqueness of the stationary subgame perfect equilibrium payoffs when the players have equal recognition probabilities. Baron and Ferejohn (1989) also show with an example that when the players have different probabilities of being selected as proposer, the equilibrium need not be unique. In particular, they construct an example with a continuum of equilibria. However, in this example all the equilibria yield the same payoffs.

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¹ See Eraslan and McLennan (2013) for a survey of the large literature building on Baron and Ferejohn (1989).

In Eraslan (2002), I extend Baron and Ferejohn's (1989) model to general q -quota agreement rules, allow the discount factors to differ across players, and show that the vector of stationary subgame perfect equilibrium payoffs is unique for general recognition probabilities.² The model I consider in this paper generalizes Eraslan (2002) by allowing risk-averse players with all players having the same concave utility functions. As in Eraslan (2002), I first establish certain monotonicity properties of the equilibrium payoffs. I show that, when the players have a common discount factor, the equilibrium payoffs are monotone non-decreasing in the recognition probabilities. Furthermore, for the case where the players have equal recognition probabilities, I show that the equilibrium payoffs are monotone non-decreasing in the discount factors.³ The proof approach I use is similar to that in Eraslan (2002), the main difference being a technical lemma (Lemma 2). I also add a step that was missing in Eraslan (2002) but was not noticed until now (see Lemma 3).

The model is a special case of Banks and Duggan (2000) who establish the existence of stationary subgame perfect equilibria when the set of alternatives is multidimensional and players are risk averse. However, they do not establish the uniqueness of the equilibrium payoffs. This model is also related to that in Harrington (1990) who considers a legislative bargaining model with risk-averse players potentially with different levels of risk aversion but with identical recognition probabilities and identical discount factors. He shows the uniqueness of equilibrium payoffs when the preferences of the players are not "too heterogeneous."

The paper is organized as follows. The next section introduces the model and characterizes the set of stationary subgame perfect equilibrium payoffs. Section 3 establishes certain monotonicity properties of the equilibrium payoffs. Section 4 proves the uniqueness of stationary subgame perfect equilibrium payoffs. Section 5 concludes.

2 The model

The agents in the set $N := \{1, \dots, n\}$ bargain over the division of a pie of size 1 according to the following protocol. At the beginning of each period until agreement is reached, a *proposer* is randomly determined. The probability that agent i is selected to be the proposer, denoted by p_i , is called i 's *recognition probability*. Let $p := (p_1, \dots, p_n)$ be the vector of recognition probabilities. Each p_i is of course non-negative, and $\sum_{i=1}^n p_i = 1$.

The proposer selects a proposal from the set $X := \{x \in [0, 1]^n : \sum_{i=1}^n x_i \leq 1\}$ of feasible allocations. An ordering of the agents is randomly determined, after which the agents each vote for or against the proposal, with each agent seeing the votes of other agents in the ordering before selecting his own vote. If at least $q \in \{1, \dots, n\}$ players, including the proposer, vote for the proposal, then the proposal is implemented, ending the game. Otherwise the process is repeated in the next period. The utility for agent i if the proposal x is implemented in period t is $\delta_t^i u(x_i)$, where

² Kalandrakis (2014) provides an alternative approach to recover the uniqueness result of Eraslan (2002) by characterizing equilibria in terms of two variables which satisfy a pair of piecewise linear equations, and Eraslan and McLennan (2013) provide an alternative proof in a more general model with arbitrary winning coalitions by showing that there is a unique connected component of equilibria all sharing the same vector of continuation payoffs.

³ As pointed out by Colin Stewart (personal communication; cf. Yıldırım 2007), this assertion is not true in Eraslan (2002). This issue is studied in detail by Kawamori (2005). The reason why the assertion is correct here is that in this paper I define equilibrium payoff as the discounted payoff of the continuation game, whereas in Eraslan (2002), I define it as the undiscounted payoff of the continuation game. Obviously there is a one-to-one relationship between the two payoffs. See the example at the end of Section 3.

$\delta = (\delta_1, \dots, \delta_n) \in (0, 1)^n$ is a vector of discount factors. I assume $u(0) = 0$, u is concave and strictly increasing. If agreement is never reached, then each agent's utility is zero.

Due to space limitations, I omit a formal definition of stationary subgame perfect (SSP) equilibrium and a formal existence result. The definition in the Appendix of Eraslan and McLennan (2013) and the existence result (Theorem 3) in Eraslan (2002) can be extended in a straightforward way to the game studied here with risk-averse players. In what follows, I define the SSP equilibrium payoffs directly.

First, note that in equilibrium, when the equilibrium SSP continuation payoff vector⁴ is $v = (v_1, \dots, v_n)$, player i rejects the offer $x \in X$ if $u(x_i) < v_i$ and accepts it if $u(x_i) \geq v_i$.⁵ Thus, given a payoff vector v , any proposal player j makes with positive probability in equilibrium can be written as $x^j(v) = (x_1^j(v), \dots, x_n^j(v))$, with $x_i^j(v) = r_{ji}(v)u^{-1}(v_i)$ for any player $i \neq j$, and $x_j^j(v) = 1 - \sum_{i \neq j} r_{ji}(v)u^{-1}(v_i)$, where $r_{ji}(v) \in \{0, 1\}$ for all i , and the vector $r^j(v) = (r_{j1}(v), \dots, r_{jn}(v))$ solves $r^j(v) \in \operatorname{argmin}_{r' \in \{0,1\}^n} \sum_{i \neq j} r'_i u^{-1}(v_i)$ subject to $\sum_{i \neq j} r'_i = q - 1$.

Note that when player j is the proposer, he is indifferent among all the proposals in the support of his equilibrium strategy. Consider now the solutions to the following problem:

$$\min_{r' \in \{0,1\}^n} \sum_{i \neq j} r'_i u^{-1}(v_i), \quad \text{subject to } \sum_{i \neq j} r'_i = q - 1. \tag{1}$$

This differs from the original problem because its minimizers correspond to mixed proposals rather than pure proposals. Clearly, if $r^j(v)$ is a solution to this problem, then one can represent it as the weighted average of the solutions to the original problem for some weights.

Thus, using the one-shot deviation principle, one can represent the equilibrium payoffs as

$$v_j = \delta_j \left[p_j u \left(1 - \sum_{i \neq j} r_{ji}(v) u^{-1}(v_i) \right) + \sum_{i \neq j} p_i r_{ij}(v) u^{-1}(v_j) \right] \tag{2}$$

for all j , where $r^j(v) = (r_{j1}(v), \dots, r_{jn}(v))$ solves problem (1) for all j . In summary, $v = (v_1, \dots, v_n)$ is an SSP equilibrium payoff vector if and only if it satisfies Equation (2) for all j .

Given an SSP payoff vector v and the corresponding matrix of offer probabilities $[r_{ij}(v)]$, let $w_i(v)$ denote the cost of the cheapest coalition (excluding the cost of his own vote) when i is the proposer, and let $\mu_i(v)$ denote the probability that player i is in the winning coalition when he is not the proposer. Formally,

$$\mu_i(v) = \sum_{j \neq i} p_j r_{ji}(v)$$

and

~~$$w_i(v) = \sum_{j \neq i} p_j r_{ij}(v) u^{-1}(v_j).$$~~

$w_i(v) = \sum_{j \neq i} p_{ij}(v) u^{-1}(v_j)$
i.e. there is no p_j in the summation.

⁴ In what follows I refer to a continuation payoff vector simply as a payoff vector.

⁵ Here I assume without loss of generality that player i accepts an offer when indifferent. Suppose to the contrary that player i rejects an offer x with $u(x_i) = v_i$ with positive probability. If there is no player j who makes an offer x with $u(x_i) = v_i$ with positive probability in equilibrium, then clearly player i 's decision when indifferent is irrelevant, and hence there is another equilibrium which is payoff equivalent to the original equilibrium in which player i accepts any offer x with $u(x_i) = v_i$ with probability one. If instead there is some player j who makes an offer x with $u(x_i) = v_i$ with positive probability, then player j can increase his payoff by slightly increasing x_i and decreasing x_j .

In equilibrium, we must have

$$v_i = \delta_i [p_i u(1 - w_i(v)) + \mu_i(v)v_i], \tag{3}$$

for all $i \in N$. Rearranging Equation (3), we can write v_i as

$$v_i = \frac{\delta_i p_i u(1 - w_i(v))}{1 - \delta_i \mu_i(v)}. \tag{4}$$

Note that $\delta_i p_i < 1 - \delta_i \mu_i(v)$ for all $i \in N$, and therefore $v_i < u(1 - w_i(v))$ for all $i \in N$. Consequently the implicit assumption I have made that there is no delay in equilibrium is self-consistent. One can also show that there is no equilibrium with delay (see Theorem 1 in Banks and Duggan 2000).

3 Monotonicity of SSP payoffs

In this section, I show that any SSP payoff vector must satisfy certain monotonicity conditions. These conditions are used in the next section to establish the uniqueness of SSP payoffs. Throughout this section I enumerate N as $\{i_1, \dots, i_n\}$ such that

$$v_{i_1} \leq \dots \leq v_{i_n}.$$

Thus, in equilibrium, the following conditions must hold. First,

$$\mu_j(v) = \begin{cases} 1 - p_j & \text{if } v_j < v_{i_q}, \\ 0 & \text{if } v_j > v_{i_q}, \end{cases} \tag{5}$$

and $\mu_j(v) \leq 1 - p_j$ if $v_i = v_{i_q}$. Second,

$$w_j(v) = \begin{cases} w_{i_q}(v) = \sum_{k=1}^{q-1} u^{-1}(v_{i_k}) & \text{if } v_j \geq v_{i_q}, \\ w_{i_q}(v) + u^{-1}(v_{i_q}) - u^{-1}(v_j) & \text{if } v_j \leq v_{i_q}. \end{cases} \tag{6}$$

Third, using conditions (5) and (6), we can write Equation (4) as

$$v_j = \begin{cases} \delta_j p_j u(1 - w_{i_q}(v)) & \text{if } v_j > v_{i_q}, \\ \frac{\delta_j p_j}{1 - \delta_j \mu_j(v)} u(1 - w_{i_q}(v)) & \text{if } v_j = v_{i_q}, \\ \frac{\delta_j p_j}{1 - \delta_j(1 - p_j)} u(1 - w_{i_q}(v) - u^{-1}(v_{i_q}) + u^{-1}(v_j)) & \text{if } v_j < v_{i_q}. \end{cases} \tag{7}$$

Proposition 1 *Let v be an SSP payoff vector and let $\{i_1, \dots, i_n\}$ be an enumeration of N such that $v_{i_1} \leq \dots \leq v_{i_n}$.*

- (i) *If $v_j \leq v_{i_q} < v_k$, then $\delta_j p_j < \delta_k p_k$.*
- (ii) *If $v_j < v_k \leq v_{i_q}$, then $\frac{\delta_j p_j}{1 - \delta_j(1 - p_j)} < \frac{\delta_k p_k}{1 - \delta_k(1 - p_k)}$.*
- (iii) *If $v_k \geq v_{i_q}$ and $\delta_j p_j \leq \delta_k p_k$, then $v_k \geq v_j$.*
- (iv) *If $v_k < v_{i_q}$ and $\frac{\delta_j p_j}{1 - \delta_j(1 - p_j)} \leq \frac{\delta_k p_k}{1 - \delta_k(1 - p_k)}$, then $v_k \geq v_j$.*

Note that, when the players have a common discount factor, $\delta_j p_j \leq \delta_k p_k$ if and only if $\frac{\delta_j p_j}{1-\delta_j(1-p_j)} \leq \frac{\delta_k p_k}{1-\delta_k(1-p_k)}$, and both of these conditions are equivalent to $p_j \leq p_k$. Thus, parts (iii) and (iv) of Theorem 1 imply that when the players have a common discount factor, the SSP payoffs are monotone non-decreasing in the recognition probabilities.

Corollary 1 *Let v be an SSP payoff vector and suppose $\delta_i = \delta$ for all $i \in N$. Then $p_j \leq p_k$ implies $v_j \leq v_k$.*

Similarly, when the players have equal recognition probabilities, $\delta_j p_j \leq \delta_k p_k$ if and only if $\frac{\delta_j p_j}{1-\delta_j(1-p_j)} < \frac{\delta_k p_k}{1-\delta_k(1-p_k)}$, and both of these conditions are equivalent to $\delta_j \leq \delta_k$. Thus, parts (iii) and (iv) of Theorem 1 imply that when the players have equal recognition probabilities, the SSP payoffs are monotone non-decreasing in the discount factors.

Corollary 2 *Let v be an SSP payoff vector and suppose $p_i = 1/n$ for all $i \in N$. Then $\delta_j \leq \delta_k$ implies $v_j \leq v_k$.*

The following example illustrates that the above monotonicity result does not hold for the undiscounted payoffs. There are three players with linear preferences, equal recognition probabilities, and the discount factors are $\delta_1 = 0.5$, $\delta_2 = 0.65$ and $\delta_3 = 0.95$. The equilibrium payoffs are $v_1 = 0.194$, $v_2 = 0.223$, and $v_3 = 0.255$.⁶ Let $\bar{v} = (\bar{v}_1, \bar{v}_2, \bar{v}_3)$ denote the undiscounted payoff vector, that is, the SSP payoff vector characterized in Eraslan (2002). Then $v_i = \delta_i \bar{v}_i$ for all i . Note that although $v_1 < v_2 < v_3$ consistent with Corollary 2, we have $\bar{v}_1 > \bar{v}_2 > \bar{v}_3$.

4 Uniqueness of SSP payoffs

In what follows, let v and \bar{v} denote two SSP equilibrium payoff vectors. Denote the corresponding disbursement vectors by $w = (w_1, \dots, w_n)$ and $\bar{w} = (\bar{w}_1, \dots, \bar{w}_n)$ respectively, and denote the corresponding inclusion probabilities by $\mu = (\mu_1, \dots, \mu_n)$ and $\bar{\mu} = (\bar{\mu}_1, \dots, \bar{\mu}_n)$. Let $\{i_1, \dots, i_n\}$ be an enumeration of N such that

$$u^{-1}(v_{i_1}) \leq \dots \leq u^{-1}(v_{i_n}),$$

and let $\{j_1, \dots, j_n\}$ be an enumeration of N such that

$$u^{-1}(\bar{v}_{j_1}) \leq \dots \leq u^{-1}(\bar{v}_{j_n}).$$

I first show that it is possible to find a player whose equilibrium payoff is equal to the q th lowest payoff under both equilibria. Furthermore, the monotonicity results I established in the previous section imply that there is some commonality in the ordering of the players according to their payoffs under the two equilibria. More precisely, it is not possible to have two players i and j with i being among the cheapest players when the equilibrium payoff vector is v but among the more expensive players when the payoff vector is \bar{v} , while the opposite is true for j .

⁶ Given these payoffs, we have $r^1(v) = (0, 1, 0)$, $r^2(v) = (1, 0, 0)$, and $r^3(v) = (1, 0, 0)$ by (1). Plugging these in (2), we obtain the payoffs we started with, confirming that these payoffs are equilibrium payoffs.

Lemma 1

- (i) *There exists $k \in N$ such that $v_k = v_{i_q}$ and $\bar{v}_k = \bar{v}_{j_q}$.*
- (ii) *Either $v_k \leq v_{i_q}$ whenever $\bar{v}_k \leq \bar{v}_{j_q}$ for all $k \in N$, or $\bar{v}_k \leq \bar{v}_{j_q}$ whenever $v_k \leq v_{i_q}$ for all $k \in N$.*

By Lemma 1, without loss of generality we can choose $i_q = j_q = q$, and assume that

$$v_i \leq v_q \quad \text{implies} \quad \bar{v}_i \leq \bar{v}_q \quad \text{for all } i \in N. \tag{*}$$

In other words, if i is not among the most expensive players when the equilibrium payoff vector is v , then he cannot be among the most expensive players when the payoff vector is \bar{v} either.

The next result shows that if player i is included as a coalition partner when the equilibrium payoff vector is v more often than he is included when the equilibrium payoff vector is \bar{v} , then there must be another player j for whom the opposite is true: player i is included as a coalition partner when the equilibrium payoff vector is v less often than he is included when the equilibrium payoff vector is \bar{v} . The fact that other players choose their coalitions partners to maximize their own payoffs in turn implies restrictions on the equilibrium payoff vectors of players i and j relative to player q under the two equilibria.

Lemma 2 (i) *For all $i \in N$, if $\mu_i > \bar{\mu}_i$, then there exists $j \in N$ such that $\mu_j < \bar{\mu}_j$,*

$$v_j \geq v_q \geq v_i \quad \text{and} \quad \bar{v}_i = \bar{v}_q \geq \bar{v}_j. \tag{8}$$

(ii) *For $i \in N$, if $\mu_i < \bar{\mu}_i$, then there exists $j \in N$ such that $\mu_j > \bar{\mu}_j$,*

$$v_j \leq v_q \leq v_i \quad \text{and} \quad \bar{v}_i \leq \bar{v}_q = \bar{v}_j. \tag{9}$$

I next show that if a player is among the most expensive players when the equilibrium payoff vector is v , then he cannot be among the cheapest players when the equilibrium payoff vector is \bar{v} .

Lemma 3 *For any $i \in N$, if $v_i > v_q$, then $\bar{v}_i \geq \bar{v}_q$.*

By assumption (*) and Lemma 3, without loss of generality, we can assume that $v_i \leq v_q$ and $\bar{v}_i \leq \bar{v}_q$ for all $i \leq q$, and $v_i \geq v_q$ and $\bar{v}_i \geq \bar{v}_q$ for all $i \geq q$. In particular, we have $w_q = \sum_{i=1}^{q-1} u^{-1}(v_i)$ and $\bar{w}_q = \sum_{i=1}^{q-1} u^{-1}(\bar{v}_i)$.

Our proof exploits the fact that if $v \neq \bar{v}$, then there must exist players i and j such that either (8) or (9) must hold. By Lemma 3, we can strengthen these expressions.

Lemma 4 *For any $i, j \in N$, with $\bar{v}_i = \bar{v}_q \geq \bar{v}_j$ and*

$$v_j \geq v_q \geq v_i, \tag{10}$$

if any of the inequalities in (10) is strict, then $\bar{v}_q = \bar{v}_j$.

Recall that a player's equilibrium payoff is determined by two endogenous factors: the cost of his coalition when he is the proposer, and the probability of being included in others' coalitions. The next result implies that the latter factor does not play a role in determining whether a player receives a higher equilibrium payoff under one equilibrium relative to his equilibrium payoff in

another equilibrium. In particular, if the cost of a player’s coalition goes down, then his payoff must go up. Furthermore, if his probability of being included in others’ coalitions also goes down, then the change in his equilibrium payoff is bounded above by the changes in his utility from the surplus he receives after paying off his coalition partners.

The remaining results also hold when we replace (v, w, μ) with $(\bar{v}, \bar{w}, \bar{\mu})$.

Lemma 5

- (i) For all $i \in N$, if $w_i \geq \bar{w}_i$ and $\mu_i \leq \bar{\mu}_i$, then $v_i \leq \bar{v}_i$. The inequality is strict if $w_i \neq \bar{w}_i$ or $\mu_i \neq \bar{\mu}_i$.
- (ii) For all $i \in N$, if $w_i \geq \bar{w}_i$ and $\mu_i \geq \bar{\mu}_i$, then $0 \leq \bar{v}_i - v_i \leq u(1 - \bar{w}_i) - u(1 - w_i)$. The inequalities are strict if $w_i \neq \bar{w}_i$ or $\mu_i \neq \bar{\mu}_i$.

The next result extends Lemma 5. In particular, the change in the equilibrium payoff of a player is bounded above by the changes in his utility from the pie he receives after paying off his coalition partners, regardless of how his probability of being included in others’ coalitions changes.

Lemma 6 For all $i \in N$, if $w_i \geq \bar{w}_i$, then $0 \leq u^{-1}(\bar{v}_i) - u^{-1}(v_i) \leq w_i - \bar{w}_i$. The inequalities are strict if and only if $w_i > \bar{w}_i$.

Using Lemma 6, we can show that the cost of winning coalition changes in the same direction for all players.

Lemma 7 For all $i \in N$, $w_i \geq \bar{w}_i$ if $w_q \geq \bar{w}_q$. The inequality is strict if and only if $w_q > \bar{w}_q$.

We are now ready to show that $v = \bar{v}$. It suffices to prove that $\bar{w}_q = w_q$. By Lemma 7, this implies that $\bar{w}_i = w_i$ for all i , and by Lemma 6 we have $\bar{v}_i = v_i$ for all i .

Suppose to the contrary that $\bar{w}_q \neq w_q$, and without loss of generality assume that $\bar{w}_q > w_q$. Then, by Lemma 7, $\bar{w}_i > w_i$ for all $i \in N$. In particular, for $i = 1, \dots, q - 1$, $\bar{w}_i > w_i$. But by Lemma 6, this implies that $\bar{v}_i < v_i$ for all $i = 1, \dots, q - 1$, which in turn implies that $\bar{w}_q = \sum_{i=1}^{q-1} u^{-1}(\bar{v}_i) < \sum_{i=1}^{q-1} u^{-1}(v_i) = w_q$. This contradiction proves the desired result and establishes the main result of the paper, with which we end this section.

Theorem 1 If v and \bar{v} are two SSP equilibrium payoffs, then $v = \bar{v}$.

5 Concluding remarks

In this paper I generalized the uniqueness result in Eraslan (2002) to a model with risk-averse players who share identical utility functions, and am thereby led to two directions for future research. One is to allow general winning coalitions as in Eraslan and McLennan (2013), and the other is to allow the players to have different utility functions. I hope to convince Andy to pursue these directions together.

Appendix

Lemma A.1 For any x, y, x' , and y' , with $0 < x < y$ and $0 < x' < y'$, if $x < x'$ and $y < y'$, then $\frac{u(x)-u(y)}{x-y} \geq \frac{u(x')-u(y')}{x'-y'}$.

PROOF: Let $a = u(x') - u(y)$, $b = x' - y$, $c = u(x') - u(y')$, $d = x' - y'$, and $k = f(y') - [\lambda u(y) + (1 - \lambda)u(x')]$, where $\lambda = \frac{x'-y'}{x'-y}$. Note that since u is concave, $k \geq 0$, and so $\frac{c}{d} \leq \frac{c+k}{d}$. Also note that $\frac{a}{b} = \frac{c+k}{d}$. Hence $\frac{u(x')-u(y)}{x'-y} = \frac{a}{b} \geq \frac{c}{d} =$

$\frac{u(x')-u(y')}{x'-y'}$. The proof is complete since $\frac{u(x)-u(y)}{x-y} \geq \frac{u(x')-u(y)}{x'-y}$. This is because $\frac{x-y}{x'-y}x' + (1 - \frac{x-y}{x'-y})y = x$, and $\frac{x-y}{x'-y} \in (0, 1)$ implies $u(x) \geq \frac{x-y}{x'-y}u(x') + (1 - \frac{x-y}{x'-y})u(y)$. and $x' \geq x$ □

Lemma A.2 For any $x, y, x',$ and $y',$ if $0 \leq y - x < y' - x'$, then $u^{-1}(y) - u^{-1}(x) < u^{-1}(y') - u^{-1}(x')$.

PROOF: Since u is concave and strictly increasing, its inverse is strictly convex. Thus

$$\frac{u^{-1}(y) - u^{-1}(x)}{y - x} < \frac{u^{-1}(y') - u^{-1}(x')}{y' - x'}$$

which implies that

$$\frac{u^{-1}(y) - u^{-1}(x)}{u^{-1}(y') - u^{-1}(x')} < \frac{y - x}{y' - x'} < 1.$$

□

Lemma A.3 Let v be an SSP payoff vector and let $\{i_1, \dots, i_n\}$ be an enumeration of N such that $v_{i_1} \leq \dots \leq v_{i_n}$.

(i) If $v_j \leq v_k$, then

$$u(1 - w_{i_q} - u^{-1}(v_{i_q}) + u^{-1}(v_k)) - u(1 - w_{i_q} - u^{-1}(v_{i_q}) + u^{-1}(v_j)) \leq v_k - v_j.$$

(ii) If $v_j < v_{i_q} \leq v_k$, then

$$v_k > \frac{p_j \delta_j}{1 - \delta_j + p_j \delta_j} u(1 - w_{i_q}). \tag{A1}$$

PROOF: (i) Since u is concave and strictly increasing, by Lemma A.1, for any $K > 0$ we have

$$u(K + u^{-1}(v_k)) - u(K + u^{-1}(v_j)) \leq u(u^{-1}(v_k)) - u(u^{-1}(v_j)) \leq v_k - v_j.$$

By (7), we have $1 - w_{i_q} - u^{-1}(v_{i_q}) > 0$ since $\delta_j p_j \leq 1 - \delta_j \mu_j(v)$. Thus the result follows. (ii) If not, we obtain a contradiction since

$$\begin{aligned} v_k - v_j &\leq \frac{p_j \delta_j}{1 - \delta_j + p_j \delta_j} [u(1 - w_{i_q}) - u(1 - w_{i_q} - u^{-1}(v_{i_q}) + u^{-1}(v_j))] \\ &\leq \frac{p_j \delta_j}{1 - \delta_j + p_j \delta_j} [u(1 - w_{i_q} - u^{-1}(v_{i_q}) + u^{-1}(v_k)) - u(1 - w_{i_q} - u^{-1}(v_{i_q}) + u^{-1}(v_j))] \\ &\leq \frac{p_j \delta_j}{1 - \delta_j + p_j \delta_j} [v_k - v_j] \\ &< v_k - v_j, \end{aligned}$$

where the first inequality follows from (7), the second inequality holds because u is strictly increasing, and the third inequality follows from part (i). □

Proof of Proposition 1

(i) If $\delta_j p_j \geq \delta_k p_k$, then we obtain a contradiction since

$$\begin{aligned} v_k - v_j &= \delta_k p_k u(1 - w_{i_q}) - \frac{\delta_j p_j}{1 - \delta_j \mu_j} u(1 - w_{i_q} - u^{-1}(v_{i_q}) + u^{-1}(v_j)) \\ &\leq \delta_k p_k [u(1 - w_{i_q}) - u(1 - w_{i_q} - u^{-1}(v_{i_q}) + u^{-1}(v_j))] \\ &< \delta_k p_k [u(1 - w_{i_q} + u^{-1}(v_{i_q}) + u^{-1}(v_k)) - u(1 - w_{i_q} - u^{-1}(v_{i_q}) + u^{-1}(v_j))] \\ &\leq \delta_k p_k [v_k - v_j] \\ &< v_k - v_j, \end{aligned}$$

where the first line follows from (7), the second line follows from (5) and the assumption that $\delta_j p_j \geq \delta_k p_k$, the third line follows from the fact that u is strictly increasing, and the fourth line follows from part (i) of Lemma 3.

(ii) If $\frac{\delta_j p_j}{1-\delta_j(1-p_j)} \geq \frac{\delta_k p_k}{1-\delta_k(1-p_k)}$, then we obtain a contradiction since

$$\begin{aligned} v_k - v_j &= \frac{\delta_k p_k}{1-\delta_k \mu_k} u(1-w_{i_q} - u^{-1}(v_{i_q}) + u^{-1}(v_k)) - \frac{\delta_j p_j}{1-\delta_j(1-p_j)} u(1-w_{i_q} - u^{-1}(v_{i_q}) + u^{-1}(v_j)) \\ &\leq \frac{\delta_j p_j}{1-\delta_j(1-p_j)} [u(1-w_{i_q} - u^{-1}(v_{i_q}) + u^{-1}(v_k)) - u(1-w_{i_q} - u^{-1}(v_{i_q}) + u^{-1}(v_j))] \\ &\leq \frac{\delta_j p_j}{1-\delta_j(1-p_j)} [v_k - v_j] \\ &< v_k - v_j, \end{aligned}$$

where the first line follows from (7), the second line follows from the fact that $\mu_j = 1 - p_j$ and $\mu_k \leq 1 - p_k$, and the assumption that $\frac{\delta_j p_j}{1-\delta_j(1-p_j)} \geq \frac{\delta_k p_k}{1-\delta_k(1-p_k)}$, and the third line follows from part (i) of Lemma 3.

(iii) If $v_k < v_j$, then $v_j > v_{i_q}$, and by (7) we have

$$v_j = \delta_j p_j u(1-w_{i_q}) > v_k = \frac{\delta_k p_k}{1-\delta_k \mu_k} u(1-w_{i_q}) \geq p_k \delta_k u(1-w_{i_q}),$$

contradicting the assumption that $\delta_j p_j \leq \delta_k p_k$.

(iv) Suppose to the contrary that $v_k < v_j$. If $v_{i_q} \geq v_j$, then by part (ii) of the proposition, $\frac{\delta_j p_j}{1-\delta_j(1-p_j)} > \frac{\delta_k p_k}{1-\delta_k(1-p_k)}$, which is a contradiction. If instead, $v_j > v_{i_q}$, then we have

$$\frac{\delta_j p_j}{1-\delta_j(1-p_j)} u(1-w_{i_q}) > \delta_j p_j u(1-w_{i_q}) = v_j > \frac{p_k \delta_k}{1-\delta_k + p_k \delta_k} u(1-w_{i_q})$$

where the equality is by (7), and the last inequality follows from part (ii) of Lemma 3. This again contradicts the assumption that $\frac{\delta_j p_j}{1-\delta_j(1-p_j)} \leq \frac{\delta_k p_k}{1-\delta_k(1-p_k)}$. □

Proof of Lemma 1

We first prove part (ii). Suppose not. Then there exist k and k' such that $v_k \leq v_{i_q} < v_{k'}$ and $\bar{v}_k \leq \bar{v}_{j_q} < \bar{v}_{k'}$. The first set of inequalities implies $\delta_{k'} p_{k'} > \delta_k p_k$, and the second set of inequalities implies $\delta_{k'} p_{k'} < \delta_k p_k$ by part (i) of Proposition 1, leading to a contradiction.

To prove part (i), first partition N as

$$\begin{aligned} N_1 &= \{k \in N : v_k < v_{i_q}\}, \\ N_2 &= \{k \in N : v_k = v_{i_q}\}, \\ N_3 &= \{k \in N : v_k > v_{i_q}\}. \end{aligned}$$

Similarly, define \bar{N}_1, \bar{N}_2 and \bar{N}_3 by replacing v with \bar{v} and i_q with j_q .

By part (ii), assume without loss of generality that, $\bar{v}_k \leq \bar{v}_{j_q}$ whenever $v_k \leq v_{i_q}$ for all $k \in N$, which implies $N_1 \cup N_2 \subseteq \bar{N}_1 \cup \bar{N}_2$. Suppose the assertion of part (i) of the lemma is not true, that is, $N_2 \cap \bar{N}_2 = \emptyset$. Then it must be the case that $N_2 \subseteq \bar{N}_1$. In particular, $\bar{v}_{i_q} < \bar{v}_{j_q}$.

Note that $N_1 \cap \bar{N}_2 = \emptyset$. If this were not the case, then there would exist $k \in N$ such that $v_k < v_{i_q}$ and $\bar{v}_k = \bar{v}_{j_q} > \bar{v}_{i_q}$. But the former inequality implies $\frac{\delta_k p_k}{1-\delta_k(1-p_k)} < \frac{\delta_{i_q} p_{i_q}}{1-\delta_{i_q}(1-p_{i_q})}$ and the latter set of inequalities implies $\frac{\delta_k p_k}{1-\delta_k(1-p_k)} > \frac{\delta_{i_q} p_{i_q}}{1-\delta_{i_q}(1-p_{i_q})}$ by part (ii) of Proposition 1. Hence $N_1 \cap \bar{N}_2 = \emptyset$.

Since $(N_1 \cup N_2) \cap \bar{N}_2 = \emptyset$ and $N_1 \cup N_2 \subseteq \bar{N}_1 \cup \bar{N}_2$, it must be the case that $N_1 \cup N_2 \subseteq \bar{N}_1$. But, by definition, $\#(N_1 \cup N_2) \geq q$ and $\#\bar{N}_1 \leq q - 1$, leading to a contradiction. □

Proof of Lemma 2

Let $i \in N$ be such that $\mu_i > \bar{\mu}_i$. Then there must exist at least one player $j \in N$ with $\mu_j < \bar{\mu}_j$, for otherwise at least one player is not making an optimal coalition choice when he is the proposer. Since $\mu_i > \bar{\mu}_i \geq 0$, it must be the case that $v_i \leq v_q$ by (5). Also $1 - p_i \geq \mu_i > \bar{\mu}_i$ implies $\bar{v}_i \geq \bar{v}_q$ by (5) again. Since $v_i \leq v_q$, we also have $\bar{v}_i \leq \bar{v}_q$ by (8), and therefore $\bar{v}_i = \bar{v}_q$. By a similar argument, we obtain $v_j \geq v_q$ and $\bar{v}_j \leq \bar{v}_q$. The proof of part (ii) is analogous. □

Proof of Lemma 3

Suppose to the contrary that there exists $i \in N$ with $v_i > v_q$ and $\bar{v}_i < \bar{v}_q$. Then $\mu_i = 0 < \bar{\mu}_i = 1 - p_i$. We will show that $\bar{\mu}_j \geq \mu_j$ for all $j \in N$ with $v_j \leq v_q$, which is a contradiction by Lemma 2.

For any j with $\bar{v}_j < \bar{v}_q$, this follows immediately since $\bar{\mu}_j = 1 - p_j \geq \mu_j$. For any j with $\bar{v}_j > \bar{v}_q$, we have $v_j > v_q$ by (*), and therefore $\bar{\mu}_j \geq 0 = \mu_j$. Thus, it remains to show that $\bar{\mu}_j \geq \mu_j$ for any j with $\bar{v}_j = \bar{v}_q$ and $v_j \leq v_q$.

Note that $\delta_i p_i > \frac{\delta_j p_j}{1 - \delta_j \mu_j}$ for any j with $v_j \leq v_q$. This follows by part (ii) of Lemma 3 and (5) if $v_j < v_q$, and from (7) if $v_j = v_q$. Likewise, $\delta_j p_j > \frac{\delta_i p_i}{1 - \delta_i \mu_i}$ for any j with $\bar{v}_j = \bar{v}_q$ by part (ii) of Lemma 3 and (5). Now since $\bar{v}_q > \bar{v}_i$, for any j with $\bar{v}_j = \bar{v}_q$ and $v_j \leq v_q$ we must have

$$\frac{\delta_j p_j}{1 - \delta_j \bar{\mu}_j} > \delta_j p_j > \frac{\delta_i p_i}{1 - \delta_i (1 - p_i)} > \delta_i p_i > \frac{\delta_j p_j}{1 - \delta_j \mu_j},$$

implying that $\bar{\mu}_j > \mu_j$. □

Proof of Lemma 4

If $v_j > v_q$, then by Lemma 3 we must have $\bar{v}_j \geq \bar{v}_q$, and so $\bar{v}_q = \bar{v}_j$. Suppose now $v_j \geq v_q > v_i$. If $\bar{v}_q > \bar{v}_j$, then we have a contradiction by part (ii) of Proposition 1. Thus, $\bar{v}_q = \bar{v}_j$. □

Proof of Lemma 5

As stated in the main text, the results also hold when we replace (v, w, μ) with $(\bar{v}, \bar{w}, \bar{\mu})$. The proofs of these alternative statements are analogous, and we omit them to avoid repetition.

(i) By (4),

$$v_i = \frac{\delta_i p_i u(1 - w_i)}{1 - \delta_i \mu_i} \leq \frac{\delta_i p_i u(1 - w_i)}{1 - \delta_i \bar{\mu}_i} \leq \frac{\delta_i p_i u(1 - \bar{w}_i)}{1 - \delta_i \bar{\mu}_i} = \bar{v}_i.$$

The proof follows immediately by noting that at least one of the inequalities in the above expression is strict if $w_i > \bar{w}_i$ or $\mu_i < \bar{\mu}_i$.

(ii) First we show that $\bar{v}_i - v_i \leq u(1 - \bar{w}_i) - u(1 - w_i)$ and the inequality is strict if $w_i \neq \bar{w}_i$ or $\mu_i \neq \bar{\mu}_i$. If $v_i > \bar{v}_i$, there is nothing to prove, and so suppose $v_i \leq \bar{v}_i$. Then, by (4),

$$\begin{aligned} \bar{v}_i - v_i &= \frac{\delta_i p_i u(1 - \bar{w}_i)}{1 - \delta_i \bar{\mu}_i} - \frac{\delta_i p_i u(1 - w_i)}{1 - \delta_i \mu_i} \leq \frac{\delta_i p_i [u(1 - \bar{w}_i) - u(1 - w_i)]}{1 - \delta_i \mu_i} \\ &\leq u(1 - \bar{w}_i) - u(1 - w_i), \end{aligned}$$

where the first inequality follows from the fact that $\bar{\mu}_i \leq \mu_i$, and the second inequality follows from the fact that $\frac{\delta_i p_i}{1 - \delta_i \bar{\mu}_i} < 1$. Note that the first inequality is strict if $\mu_i \neq \bar{\mu}_i$ and the second inequality is strict if $w_i \neq \bar{w}_i$.

Next we show that $v_i \leq \bar{v}_i$ and the inequality is strict if $w_i \neq \bar{w}_i$ or $\mu_i \neq \bar{\mu}_i$. If $\mu_i = \bar{\mu}_i$, then $v_i \leq \bar{v}_i$ with strict inequality if $w_i > \bar{w}_i$ by (4), so suppose $\mu_i > \bar{\mu}_i$. By Lemma 2 there exists a player j such that $\mu_j < \bar{\mu}_j$ and (8) hold. If $v_j < \bar{v}_j$, then $v_i \leq v_j < \bar{v}_j \leq \bar{v}_i$ and the result follows, so assume $v_j \geq \bar{v}_j$. Then it must be the case that $w_j \leq \bar{w}_j$, because otherwise we have $v_j < \bar{v}_j$ by part (i). This implies $v_j - \bar{v}_j < u(1 - w_j) - u(1 - \bar{w}_j)$ from what we showed in the first part of this proof. By **Lemma A.2**, this in turn implies that $u^{-1}(v_j) - u^{-1}(\bar{v}_j) < \bar{w}_j - w_j$.

By assumption (*) and Lemma 4, there are two possible cases.

Case 1: $\bar{v}_i = \bar{v}_q \geq \bar{v}_j$ and $v_j = v_q = v_i$. By (6), $w_j = w_q = w_q + u^{-1}(v_q) - u^{-1}(v_j)$ and $\bar{w}_j = \bar{w}_q + u^{-1}(\bar{v}_q) - u^{-1}(\bar{v}_j)$. Hence, $\bar{w}_j - w_j = (\bar{w}_q - w_q) + u^{-1}(\bar{v}_q) - u^{-1}(v_q) - u^{-1}(\bar{v}_j) + u^{-1}(v_j)$, and therefore

$$u^{-1}(\bar{v}_q) - u^{-1}(v_q) = [(w_j - w_q) + (u^{-1}(\bar{v}_j) - u^{-1}(v_j))] + (w_q - \bar{w}_q).$$

Note that $w_q - \bar{w}_q$ is non-negative since $w_q = w_i \geq \bar{w}_i = \bar{w}_q$. Since the term in the brackets is strictly positive and u^{-1} is strictly increasing, the result follows by noting that $v_q = v_i$ and $\bar{v}_i = \bar{v}_q$.

Case 2: $\bar{v}_i = \bar{v}_q = \bar{v}_j$ and $v_j \geq v_q \geq v_i$. By (6), $w_i = w_q + u^{-1}(v_q) - u^{-1}(v_i)$ and $\bar{w}_i = \bar{w}_q + u^{-1}(\bar{v}_q) - u^{-1}(\bar{v}_i)$. Since $w_i \geq \bar{w}_i$, we have $u^{-1}(\bar{v}_i) - u^{-1}(v_i) \geq (\bar{w}_q - w_q) - (u^{-1}(v_q) - u^{-1}(\bar{v}_q))$. Note that $u^{-1}(v_q) - u^{-1}(\bar{v}_q) \leq u^{-1}(v_j) - u^{-1}(\bar{v}_j) < \bar{w}_j - w_j = \bar{w}_q - w_q$. Thus $\bar{v}_i > v_i$. □

Proof of Lemma 6

As stated in the main text, the result also holds when we replace (v, w, μ) with $(\bar{v}, \bar{w}, \bar{\mu})$. The proof of the alternative statement is analogous, and we omit it to avoid repetition.

By parts (i) and (ii) of Lemma 5 and the strict monotonicity of u , we have $0 \leq u^{-1}(\bar{v}_i) - u^{-1}(v_i)$, with strict inequality if $w_i \neq \bar{w}_i$.

If $\mu_i \geq \bar{\mu}_i$, then $\bar{v}_i - v_i \leq u(1 - \bar{w}_i) - u(1 - w_i)$ by part (ii) of Lemma 5, with strict inequality if $w_i \neq \bar{w}_i$. By Lemma 2, this implies $u^{-1}(\bar{v}_i) - u^{-1}(v_i) \leq w_i - \bar{w}_i$, so assume $\mu_i < \bar{\mu}_i$ in the rest of the proof. Since $\mu_i < \bar{\mu}_i$, by Lemma 2 there exists a player j such that $\mu_j > \bar{\mu}_j$. By assumption (*) and Lemma 4, there are two possible cases.

Case 1: $\bar{v}_j = \bar{v}_q \geq \bar{v}_i$ and $v_i = v_q = v_j$. By (6), $w_j = w_q = w_i$ and $\bar{w}_i \geq \bar{w}_q = \bar{w}_j$, and

$$w_i - \bar{w}_i = [(w_q - \bar{w}_q) - (u^{-1}(\bar{v}_q) - u^{-1}(v_q))] + (u^{-1}(\bar{v}_i) - u^{-1}(v_i)). \quad (A2)$$

Since $w_j = w_i \geq \bar{w}_i \geq \bar{w}_j$ and $\mu_j > \bar{\mu}_j$, by part (ii) of Lemma 5, we have $0 < \bar{v}_j - v_j < u(1 - \bar{w}_j) - u(1 - w_j)$. By Lemma 2, this implies $u^{-1}(\bar{v}_j) - u^{-1}(v_j) < w_j - \bar{w}_j$. Since u is strictly increasing, we have $u^{-1}(\bar{v}_j) - u^{-1}(v_j) \geq u^{-1}(\bar{v}_q) - u^{-1}(v_q)$. Note also that $w_j - \bar{w}_j = w_q - \bar{w}_q$. Thus the term in brackets in Equation (A2) is strictly positive, and therefore we have $u^{-1}(\bar{v}_i) - u^{-1}(v_i) < w_i - \bar{w}_i$.

Case 2: $\bar{v}_j = \bar{v}_q = \bar{v}_i$ and $v_i \geq v_q \geq v_j$. By (6), we have $w_j \geq w_q = w_i$ and $\bar{w}_i = \bar{w}_j = \bar{w}_q$. Then $w_j \geq \bar{w}_j$. Since $\mu_j > \bar{\mu}_j$, we have $0 < \bar{v}_j - v_j < u(1 - \bar{w}_j) - u(1 - w_j)$ by part (ii) of Lemma 5. By Lemma 2, this implies $u^{-1}(\bar{v}_j) - u^{-1}(v_j) < w_j - \bar{w}_j$. Using (6) again, we obtain $w_j = w_q + \delta_q v_q - \delta_j v_j \leq w_i + \delta_i v_i - \delta_j v_j$ and $\bar{w}_j = \bar{w}_q + u^{-1}(\bar{v}_q) - u^{-1}(\bar{v}_j) = \bar{w}_i + u^{-1}(\bar{v}_i) - u^{-1}(\bar{v}_j)$. Hence,

$$w_i - \bar{w}_i \geq [(w_j - \bar{w}_j) - (u^{-1}(\bar{v}_j) - u^{-1}(v_j))] + (u^{-1}(\bar{v}_i) - u^{-1}(v_i)).$$

The result follows by noting that the term in the brackets is strictly positive.

To see that the inequalities are strict only if $w_i > \bar{w}_i$, notice that, if $w_i = \bar{w}_i$, then $0 \leq u^{-1}(\bar{v}_i) - u^{-1}(v_i) \leq w_i - \bar{w}_i = 0$, which in turn implies that neither inequality can be strict. \square

Proof of Lemma 7

As stated in the main text, the results also hold when we replace (v, w, μ) with $(\bar{v}, \bar{w}, \bar{\mu})$. The proofs of these alternative statements are analogous, and we omit them to avoid repetition.

Recall that we assumed without loss of generality that $v_i \geq v_q$ and $\bar{v}_i \geq \bar{v}_i$ for all $i \geq q$ and $v_i \leq v_q$, and $\bar{v}_i \leq \bar{v}_i$ for all $i \leq q$. Thus, the proof follows immediately by (6) if $i \geq q$. So let $i < q$. Then $v_i \leq v_q$ and $\bar{v}_i \leq \bar{v}_q$ and hence, from (6),

$$\begin{aligned} w_i - \bar{w}_i &= [(w_q - \bar{w}_q) - (u^{-1}(\bar{v}_q) - u^{-1}(v_q))] + (u^{-1}(\bar{v}_i) - u^{-1}(v_i)) \\ &\geq (u^{-1}(\bar{v}_i) - u^{-1}(v_i)), \end{aligned} \quad (A3)$$

where the inequality follows from Lemma 6, and is strict if and only if $w_q > \bar{w}_q$ also by Lemma 6.

Now if $w_i < \bar{w}_i$, then $0 < u^{-1}(v_i) - u^{-1}(\bar{v}_i) < \bar{w}_i - w_i$ by Lemma 6, contradicting (A3). Hence $w_i \geq \bar{w}_i$, and the inequality is strict if $w_q > \bar{w}_q$ by (A3). Finally, if $w_q = \bar{w}_q$, then $\bar{v}_q = v_q$ by Lemma 6 and hence, by (A3), $w_i - \bar{w}_i = u^{-1}(\bar{v}_i) - u^{-1}(v_i)$. By Lemma 6 this is possible only if $w_i = \bar{w}_i$. \square

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